The Newton polytope of the Kronecker product

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Saturated Newton polytope

A multivariate polynomial with nonnegative coefficients $f(x_1, \ldots, x_k) = \sum_\alpha c_\alpha x^\alpha$ has a *saturated Newton polytope* (SNP) if the set of points $M_k(f) := \{(\alpha_1, \cdots, \alpha_k) : c_\alpha > 0\}$ coincides with its convex hull in $\mathbb{Z}^k$.

The Kronecker product

The *Kronecker coefficients* of S_n, denoted by $g(\lambda, \mu, \nu)$, give the multiplicities of one irreducible Specht module in the tensor product of the other two, namely

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}.$$

The *Kronecker product* $*$ of symmetric functions is defined on the Schur basis as

$$s_\lambda * s_\mu := \sum_\nu g(\lambda, \mu, \nu) s_\nu,$$

and extended by linearity. It is equivalent to the inner product of S_n characters under the characteristic map.

SNP for the Kronecker product

Conjecture 1 (Monical-Tokcan-Yong, 2019, [3])

The Kronecker product $s_\lambda * s_\mu$ has a saturated Newton polytope.

We prove this conjecture for partitions of lengths 2 and 3:

Theorem 1 (Panova-Zhao, 2023, [4])

Let $\lambda, \mu \vdash n$ with $\ell(\lambda) \le 2, \ell(\mu) \le 3$, and $\mu_1 \ge \lambda_1$ then $s_\lambda * s_\mu(x_1, \ldots, x_k)$ has a saturated Newton polytope for every k.

Proof idea: The Kronecker product in these cases contains a term s_ν with ν dominating all other partitions appearing. Thus the Newton polytope consists of all integer points $(a_1, \ldots, a_k)$, such that $sort(a_1, \ldots, a_k) \preceq \nu$ in the dominance order.

Example

However, we cannot expect to have unique maximal terms in general:

$s_{(6,6)} * s_{(8,2,1,1)} = s_{(4,4,2,1,1)} + s_{(4,4,3,1)} + s_{(5,3,1,1,1,1)} + s_{(5,3,2,1,1)} + s_{(5,3,2,2)} + s_{(5,3,3,1)} + 2s_{(5,4,1,1,1)} + 3s_{(5,4,2,1)} + s_{(5,4,3)} + s_{(5,5,1,1)} + 2s_{(5,5,2)} + s_{(6,2,2,1,1)} + 2s_{(6,3,1,1,1)} + 3s_{(6,3,2,1)} + s_{(6,3,3)} + 4s_{(6,4,1,1)} + 2s_{(6,4,2)} + 2s_{(6,5,1)} + s_{(7,2,1,1,1)} + s_{(7,2,2,1)} + 2s_{(7,3,1,1)} + 2s_{(7,3,2)} + 2s_{(7,4,1)} + s_{(7,5)} + s_{(8,2,1,1)} + s_{(8,3,1)}$.

In this product, $(7, 5)$ and $(8, 3, 1)$ are incomparable maximal.

The first case when there is no such dominant partition is covered in the following.

Theorem 2 (Two and three-row partitions, [4])

Let $\lambda, \mu \vdash n$ with $\ell(\lambda) \le 2$ and $\ell(\mu) \le 3$. Then $s_\lambda * s_\mu(x_1, x_2, x_3)$ has a saturated Newton polytope.

The limit case

Theorem 3 (The limit case, [4])

Let λ, μ be partitions of the same size and $k \in \mathbb{N}$. Then the set of points

$$\bigcup_{p=1}^{\infty} \frac{1}{p} M_k(s_{p\lambda} * s_{p\mu})$$

is a convex subset of $\mathbb{Q}^k$.

Conjecture 1 holds in the "limit".

Monomial expansion via multi-LR coefficients

The *multi-LR coefficients* can be defined recursively as

$$c^\lambda_{\nu^1\nu^2\ldots} := \langle s_\lambda, s_{\nu^1} s_{\nu^2} \cdots \rangle = \sum_{\tau^1,\ldots} c^\lambda_{\nu^1\tau^1} c^{\tau^1}_{\nu^2\tau^2} \cdots .$$

Coefficient of x^{a} in $s_\lambda * s_\mu$, where $\mathsf{a} := (a_1, a_2, \ldots)$:

$$\langle s_\lambda(y) * s_\mu(z), h_{\mathsf{a}}[yz] \rangle = \sum_{\alpha^i \vdash a_i, i=1,\ldots} c^\lambda_{\alpha^1\alpha^2\ldots} c^\mu_{\alpha^1\alpha^2\ldots}$$

$$P(\mu; \mathsf{a}) := \{(\alpha^1, \alpha^2, \cdots, \alpha^k) \in \mathbb{Z}_{\ge 0}^{\ell(\mu)k} : c^\mu_{\alpha^1\alpha^2\ldots} > 0 \text{ and } |\alpha^i| = a_i \text{ for } i = 1, \ldots, k\}.$$

Then, the set of monomial degrees $\mathsf{a} = (a_1, \ldots, a_k)$ appearing in $s_\lambda * s_\mu$ is given as

$$M_k(s_\lambda * s_\mu) = \{\mathsf{a} \in \mathbb{Z}^k_{\ge 0} : P(\lambda; \mathsf{a}) \cap P(\mu; \mathsf{a}) \ne \emptyset\}.$$

Horn inequalities for multi-LR's

Theorem 3 (van Leeuwen, [5])

Let λ, μ, ν be partitions such that $|\lambda| = |\mu| + |\nu|$. Then $c^\lambda_{\mu,\nu} = \langle s_\lambda, s_{\mu\diamond\nu} \rangle$, where $\mu \diamond \nu$ denotes the skew shape $(\nu_1^{\ell(\mu)} + \mu, \nu)/\nu$.

For a subset $I = \{i_1 < i_2 < \cdots < i_s\} \subset [r]$, let $\rho(I)$ denote the partition $\rho(I) := (i_s - s, \ldots, i_2 - 2, i_1 - 1)$. A triple of subsets $I, J, K \subset [r]$ is *LR-consistent* if they have the same cardinality s and $c^{\rho(I)}_{\rho(J),\rho(K)} = 1$.

Theorem 4 (Zelevinsky, Klyachko, Knutson-Tao, [6, 1, 2])

Let $\lambda, \mu, \nu \in \mathbb{N}^r$ with weakly decreasing component. Then $c^\lambda_{\mu,\nu} > 0$ if and only if $|\lambda| = |\mu| + |\nu|$ and $\sum_{i\in I} \lambda_i \le \sum_{j \in J} \mu_j + \sum_{k\in K} \nu_k$ for all LR-consistent triples $I, J, K \subset [r]$.

Corollary 4.1

Let $\ell(\mu) = \ell$ and $\mathsf{a} = (a_1, \ldots, a_k)$. Then

$$\begin{aligned} P(\mu; \mathsf{a}) = \{(\alpha^1, \ldots, \alpha^k) \in & \mathbb{Z}^{\ell k}_{\ge 0} : \\ \sum_j \alpha^i_j &= a_i, && \text{for } i \in [k]; \\ \alpha^i_j &\ge \alpha^i_{j+1}, && \text{for } j \in [\ell - 1],\ i \in [k]; \\ \sum_{(i,j)\in D(I)} (n(k-i) + \alpha^i_j) &\le \sum_{j\in J} \mu_j + \sum_{(d,r)\in D(K)} n(k-d)\}, \end{aligned}$$

where the last inequalities hold for all LR-consistent triples $I, J, K \in [\ell k]$.

The polytope for $k = 3$

Let $\ell(\lambda) = 2$ and $\ell(\mu) = 3$. Consider $\mathbf{a} = (a_1, a_2, a_3)$. Let $x := \alpha^1_1, y := \alpha^2_1, z := \alpha^3_1$. When $\mu_1 < \lambda_1$, $P(\lambda; \mathsf{a}) \cap P(\mu; \mathsf{a}) \ne \emptyset$ is equivalent to $\mathcal{Q}(\lambda, \mu, \mathsf{a}) \cap \mathbb{Z}^3 \ne \emptyset$.

$$\mathcal{Q}(\lambda, \mu, \mathsf{a}) := \Big\{ (x, y, z) \in \mathbb{R}^3 \text{ s.t. } \begin{aligned} a_1 - \min(\mu_2, \lambda_2, \tfrac{a_1}{2}) &\le x \le \min(a_1, \mu_1) \\ a_2 - \min(\mu_2, \lambda_2, \tfrac{a_2}{2}) &\le y \le \min(a_2, \mu_1) \\ a_3 - \min(\mu_2, \lambda_2, \tfrac{a_3}{2}) &\le z \le \min(a_3, \mu_1) \\ \max(\mu_3, a_1 + a_2 - \mu_1) &\le x + y \\ \max(\mu_3, a_1 + a_3 - \mu_1) &\le x + z \\ \max(\mu_3, a_2 + a_3 - \mu_1) &\le y + z \\ \lambda_1 &\le x + y + z \\ \max(\mu_2, \lambda_2) - a_1 \le -x + y + z &\le \mu_1 + \mu_2 - a_1 \\ \max(\mu_2, \lambda_2) - a_2 \le x - y + z &\le \mu_1 + \mu_2 - a_2 \\ \max(\mu_2, \lambda_2) - a_3 \le x + y - z &\le \mu_1 + \mu_2 - a_3 \end{aligned} \Big\}$$

Analyzing $\mathcal{Q}(\lambda, \mu, \mathsf{c})$, [4]

- Suppose that $\mathcal{Q}(\lambda, \mu, \mathsf{a}^i) \ne \emptyset$ for some vectors a^i, $i = 1, \ldots, 4$ and $\mathsf{c} = \sum_i t_i \mathsf{a}^i$ for some $t_i \in [0, 1]$ with $t_1 + t_2 + t_3 + t_4 = 1$. Then $\mathcal{Q}(\lambda, \mu, \mathsf{c}) \ne \emptyset$.
- [4, Theorem 2] If $\mathcal{Q}(\lambda, \mu, \mathsf{a}) \ne \emptyset$ then it has an integer point, i.e. $\mathcal{Q}(\lambda, \mu, \mathsf{a}) \cap \mathbb{Z}^3 \ne \emptyset$.

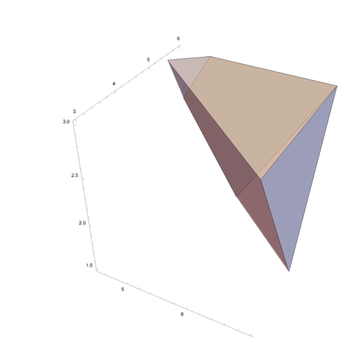

Figure 1. $\mathcal{Q}(\lambda, \mu; \mathsf{a})$ where $\lambda = (7, 6, 5)$, $\mu = (13, 5)$ and $\mathsf{a} = (9, 6, 3)$.

Kronecker positivity implications

If $g(\lambda, \mu, \nu) > 0$, then s_ν appears in $s_\lambda * s_\mu$, and so its leading monomial m_ν also appears, so $\mathcal{Q}(\lambda, \mu, \nu) \cap \mathbb{Z}^r \ne \emptyset$.

We define *mLR-consistent* triple (I, J, K) of subsets of $[1, \ldots, \ell k]$ to be the LR-consistant triples, such that $|I \cap [\ell(j-1)+1, \ldots, \ell j]| = |K \cap [\ell(j-1), \ldots, \ell j]|$ for every $j = 1, \ldots, k$.

Theorem 5 (Kronecker positivity criterion, [4])

Suppose that $g(\lambda, \mu, \nu) > 0$ and let $\ell = \min\{\ell(\mu), \ell(\nu)\}$ Then there exist nonnegative integers $\{\alpha^i_j\}_{i\in[k], j\in[\ell]}$ satisfying

$$\begin{aligned} \sum_j \alpha^i_j &= \lambda_i, && \text{for } i \in [k]; \\ \alpha^i_j &\ge \alpha^i_{j+1}, && \text{for } j \in [\ell - 1],\ i \in [k]; \\ \sum_{(i,j)\in D(I)} \alpha^i_j &\le \min\{\sum_{j\in J} \mu_j, \sum_{j\in J} \nu_j\}, && \text{for every mLR-consistent } (I, J, K). \end{aligned}$$

Corollary 5.1

Suppose that $g(\lambda, \mu, \nu) > 0$ and $\ell(\mu) = 2$, $k = \ell(\lambda)$. Then there exist nonnegative integers $y_i \in [0, \lfloor \lambda_i/2 \rfloor]$ for $i \in [k]$, such that

$$\sum_{i\in A\cup C} \lambda_i + \sum_{i\in B} y_i - \sum_{i \in C} y_i \le \min\{\sum_{j\in J} \mu_j, \sum_{j\in J} \nu_j\}$$

for all triples of mutually disjoint sets $A \sqcup B \sqcup C \subset [k]$ and $J = \{1, \ldots, r, r+2, \ldots, r+b+1\}$ or $J = \{1, \ldots, r+b-1, r+2b\}$, where $r = 2|A| + |C|$ and $b = |B|$.

Main References

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