The Monoid Representation of Upho Posets and Total Positivity

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Introduction

We establish a bijection between **locally finite colored upho posets** and **left-cancellative invertible-free monoids**. Moreover, this bijection maps **$\mathbb{N}$-graded colored upho posets** to **left-cancellative homogeneous monoids**. Furthermore, by utilizing this bijection and introducing the concept of **semi-upho posets**, we show that every **totally positive power series** with constant term 1 is the rank-generating function of some upho poset, resolving a conjecture of Gao et al.

Upper Homogeneous (Upho) Posets

A poset P is called **upper homogeneous**, abbreviated as **upho**, if each principal order filter

$$V_s := \{p \in P \mid s \leq_P p\}, s \in P$$

is isomorphic to the poset itself. A formal power series $f(x) \in 1 + x\mathbb{Z}[[x]]$ is called an **upho function** if it is the rank-generating function of a finite type $\mathbb{N}$-graded upho poset.

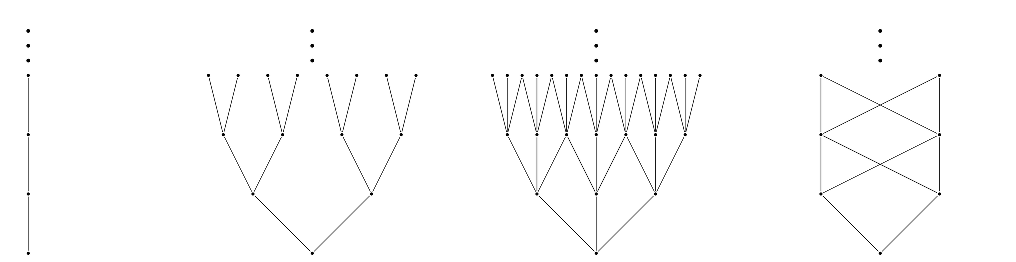

Figure 1:The Hasse diagrams of $\mathbb{N}$, the full binary tree, Stern's poset, and the bowtie poset. They are all $\mathbb{N}$-graded upho posets of finite type.

Background and Notations

The **height** of an element s in a poset P is the maximal length of chains in P with s as its maximum. A poset P is **locally finite** if every element in P has finite height.

A poset is called **tree-like** if its Hasse diagram is a tree.

In a poset P which has a unique minimum $\hat{0}$, we denote the set of **edges** of P as $\mathcal{E}_P$ and the set of **atoms** as $\mathcal{A}_P$.

A monoid M is said to be **left-cancellative** if for every $a, x, y \in M$, $ax = ay$ implies $x = y$.

A monoid M is said to be **invertible-free** if for every $x, y \in M$, $xy = e$ implies $x = y = e$, where e is the identity element.

In a monoid M, $a \in M$ is **irreducible** if it is non-invertible and is not the product of any two non-invertible elements.

An invertible-free monoid M is said to be **homogeneous** if for every element $w \in M$ satisfying

$$w = a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m,$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ are irreducible elements of M, we have $n = m$.

We abbreviate left-cancellative invertible-free as **LCIF**, and left-cancellative homogeneous as **LCH**.

Monoid Representation

A **colored upho poset** $\tilde{P}$ consists of the data (P, col_P): The poset P is an upho poset, and the color mapping $\mathrm{col}_P : \mathcal{E}_P \to \mathcal{A}_P$ satisfies the following conditions:

- For every $t \in \mathcal{A}_P$, we have $\mathrm{col}_P(\hat{0}, t) = t$;
- For every $s \in P$, there exists an isomorphism $\phi_s : V_s \xrightarrow{\sim} P$ such that for every $(u, v) \in \mathcal{E}_{V_s}$, we have
$$\mathrm{col}_P(u, v) = \mathrm{col}_P(\phi_s(u), \phi_s(v)).$$

We define a mapping $\mathcal{M}$ which maps locally finite colored upho posets to LCIF monoids by the following rule. For a given locally finite colored upho poset $\tilde{P} = (P, \mathrm{col}_P)$, the elements of $\mathcal{M}(\tilde{P})$ are the elements of P. For every $s, t \in P$, we define the multiplication st by $st := \phi_s^{-1}(t)$.

Conversely, we define a mapping $\tilde{\mathcal{P}} = (\mathcal{P}, \mathcal{C})$ which maps LCIF monoids to locally finite colored upho posets by the following rule. The elements of $\mathcal{P}(M)$ are the elements of M. The partial order $\leq_P$ in $\mathcal{P}(M)$ is defined by the left divisibility in M, that is, $a \leq_{P_M} b$ if and only if there exists $c \in M$ such that $ac = b$ in M.

Then we have the following theorem:

Theorem

The mutually inverse mappings $\mathcal{M}$ and $\tilde{\mathcal{P}}$ give a bijection between locally finite colored upho posets and left-cancellative invertible-free monoids. Moreover, this bijection maps $\mathbb{N}$-graded colored upho posets to left-cancellative homogeneous monoids, and maps finite type $\mathbb{N}$-graded colored upho posets to finitely generated left-cancellative homogeneous monoids.

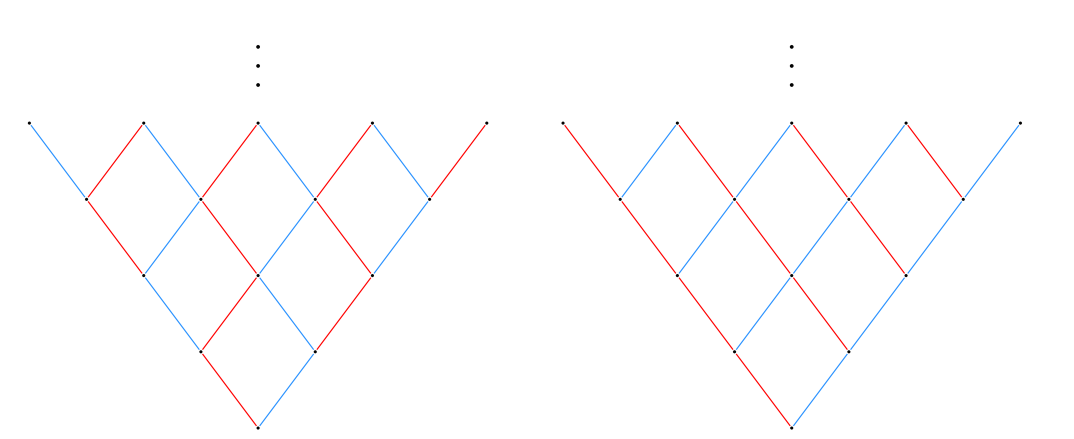

Figure 2:The left colored upho poset corresponds to the LCH monoid $\langle x_1, x_2 \mid x_1^2 = x_2^2 \rangle$, while the right corresponds to $\langle x_1, x_2 \mid x_1 x_2 = x_2 x_1 \rangle$.

The **forgetful mapping** $\mathfrak{F}$ maps a locally finite colored upho poset $\tilde{P} = (P, \mathrm{col}_P)$ to P. We define **regular upho posets** as the upho posets in **im$\mathfrak{F}$**. Figure 2 shows that $\mathfrak{F}$ is not injective, and our conjecture regarding the surjectivity of $\mathfrak{F}$ is:

Conjecture

Every locally finite upho poset is regular.

Semi-Upho Posets

We introduce **semi-upho posets**, a generalization of upho posets which have partial self-similarity.

Given posets P' and P with unique minima $\hat{0}_{P'}$ and $\hat{0}_P$ respectively, an injection $\eta : P' \hookrightarrow P$ is said to be an **induced saturated order embedding**, abbreviated as **isoembedding**, if $\eta(\hat{0}_{P'}) = \hat{0}_P$, and furthermore, for every chain C in P' with maximum a and minimum b, C is a maximal chain with maximum a and minimum b if and only if $\eta(C)$ is a maximal chain with maximum $\eta(a)$ and minimum $\eta(b)$.

A poset S is called **semi-upho** if for every $s \in S$, there exists an isoembedding $V_s \hookrightarrow S$.

Theorem

Let $g(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be a log-concave formal power series without internal zeros, then it is the rank-generating function of a tree-like semi-upho poset.

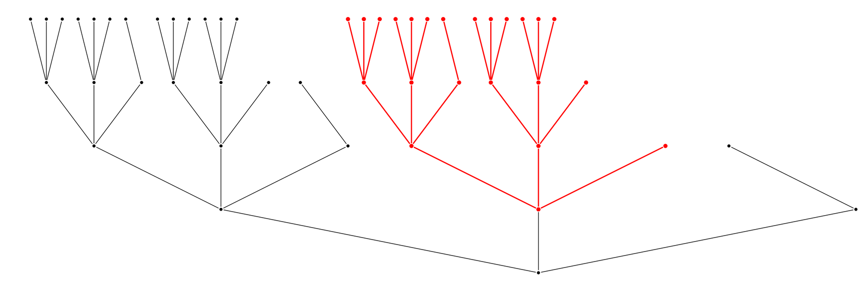

Figure 3:A tree-like semi-upho poset with a log-concave rank-generating function $f(x) = 1 + 3x + 7x^2 + 13x^3 + 26x^4$. The red part is a principal order filter that can be isoembedded into the poset itself.

The coloring and the monoid representation of upho posets can be generalized to semi-upho posets. Based on this method and some tricks with monoids, we have the following theorem.

Theorem

Let $f(x) \in 1 + x_{\geq 0}[[x]]$ be the rank-generating function of a regular upho poset P and $g(x) \in 1 + x_{\geq 0}[[x]]$ be the rank-generating function of a tree-like upho poset S, then there exists another regular upho poset R whose rank-generating function equals $f(x)g(x)$.

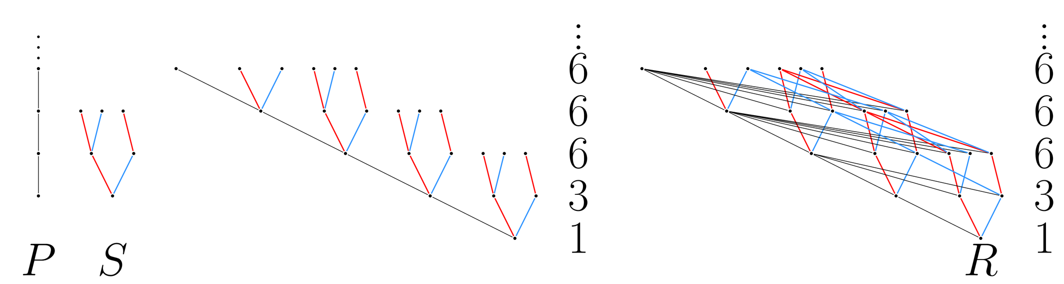

Figure 4:Construction of an upho poset R by 'convolving' a regular upho poset P and a tree-like semi-upho poset S.

By a previous theorem, we can take $g(x)$ to be a log concave formal power series.

Totally Positive Upho Functions

Our working definition of **totally positive formal power series** follows from the following theorem.

Theorem([1])

A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is totally positive if and only if $f(x)$ is of the form of $\frac{g(x)}{h(x)}$, where $g(x), h(x) \in 1 + x\mathbb{Z}[x]$ such that all the complex roots of $g(x)$ are real and negative, and all the complex roots of $h(x)$ are real and positive.

Using the theorems above, we obtain the following result.

Theorem

Let $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ be a totally positive formal power series. Then $f(x)$ is an upho function.

And finally, we show that the following conjecture of Gao et al. is a corollary of the above theorem.

The **Ehrenborg quasi-symmetric function** [2] of a finite type $\mathbb{N}$-graded poset P is defined to be $E_P := \sum_{n \geq 0} E_{P,n}$, where $E_{P,0} = 1$, and

$$E_{P,n}(x_1, x_2, \cdots, x_n) := \sum_{\substack{\hat{0} = t_0 \leq_P \cdots \leq_P t_{k-1} <_P t_k \\ \rho(t_k) = n}} \prod_{i=1}^{k} x_i^{\rho(t_i) - \rho(t_{i-1})}, \ n \geq 1.$$

Theorem([3, Conjecture 3.3])

A formal power series $f(x) \in 1 + x\mathbb{Z}_{\geq 0}[[x]]$ is the rank-generating function of an upho poset P whose Ehrenborg quasi-symmetric function is a Schur-positive symmetric function if and only if $f(x)$ is totally positive.

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