Quasipartition and planar quasipartition algebras

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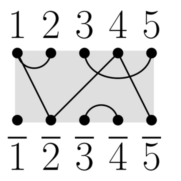
Partition algebras

The partition algebra was introduced by Martin and Jones in the 1990s.

- **Partition algebra**: $\mathsf{P}_k(n)$ is spanned by all set partitions of $[k] \cup [\overline{k}]$.
- $\mathsf{P}_k(n)$ is generated by s_i, b_i, with $1 \leq i \leq k-1$ and p_j for $1 \leq j \leq k$.
- A **block** is a set in the set partition. And a **singleton** is a block of size 1.
- **Half partition algebra**: $\mathsf{P}_{k+\frac{1}{2}}(n)$ is a subalgebra of $\mathsf{P}_{k+1}(n)$, spanned by all set partitions of $[k] \cup [\overline{k}]$ with k and $\overline{k}$ in the same block.
- $\mathsf{P}_{k+\frac{1}{2}}(n)$ is generated by s_i, b_i, with $1 \leq i \leq k$ and p_j with $1 \leq j \leq k$.
- For $n \geq 2k$, $\mathsf{P}_k(n) \cong \mathrm{End}_{S_n}(V_n^{\otimes k})$, where $V_n = \mathbb{C}^n$.
- For $n \geq 2k+1$, $\mathsf{P}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}(\mathrm{Res}^{S_n}_{S_{n-1}} V_n^{\otimes k})$.
- Bell numbers count $\dim(\mathsf{P}_k(n)) = B(2k)$ and $\dim(\mathsf{P}_{k+\frac{1}{2}}(n)) = B(2k+1)$.

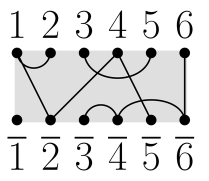
Examples

$\{\{1,2,4,\overline{2},\overline{5}\},\{3,5\},\{\overline{3},\overline{4}\},\{\overline{1}\}\}$



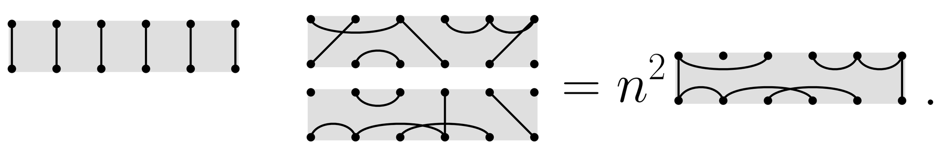
$\in \mathsf{P}_5(n)$.

$\{\{1,2,4,\overline{2},\overline{5}\},\{3,5\},\{6,\overline{3},\overline{4},\overline{6}\},\{\overline{1}\}\}$

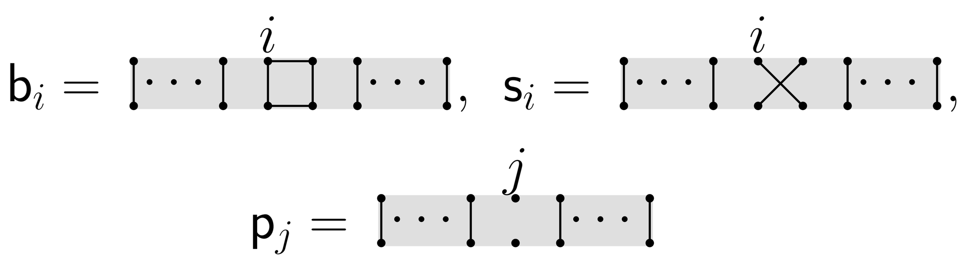


$\in \mathsf{P}_{5+\frac{1}{2}}(n) \subseteq \mathsf{P}_6(n)$.

Unit: Product:



Generators:



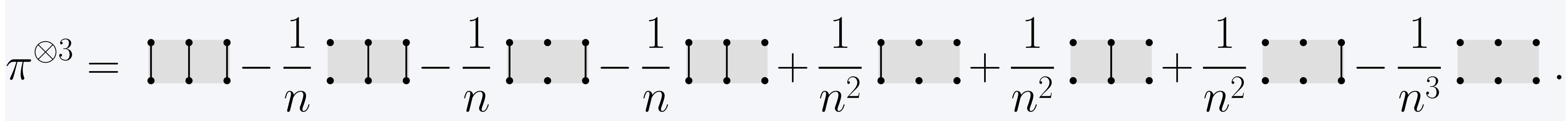
The quasipartition algebras

This algebra was introduced by Daugherty and Orellana in [1] as the centralizer algebra $\mathrm{End}_{S_n}((\mathbb{S}^{(n-1,1)})^{\otimes k})$.

We give a more general definition using the idempotent $\pi^{\otimes k}$:

- $$\pi^{\otimes k} := (\mathbf{1}^{\otimes k} - \tfrac{1}{n}\mathsf{p}_1)(\mathbf{1}^{\otimes k} - \tfrac{1}{n}\mathsf{p}_2)\cdots(\mathbf{1}^{\otimes k} - \tfrac{1}{n}\mathsf{p}_k),$$
- $$\mathsf{QP}_k(n) := \pi^{\otimes k}\mathsf{P}_k(n)\pi^{\otimes k},$$
- $$\overline{d} := \pi^{\otimes k} d \pi^{\otimes k}, \quad \text{for } d \in \mathsf{P}_k(n),$$
- $\{\overline{d} : d \in \mathsf{P}_k(n), d$ a set partition with no singletons $\}$ is a basis of $\mathsf{QP}_k(n)$ [2].

Example (Idempotent in $P_3(n)$/ Identity in $\mathsf{QP}_3(n)$)

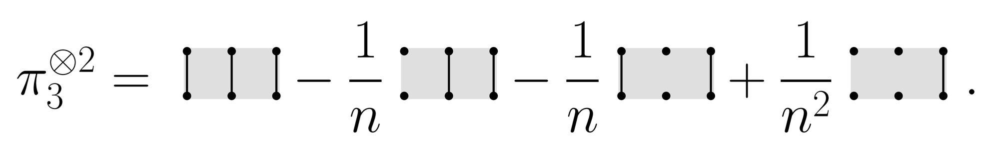


Half quasipartition algebras

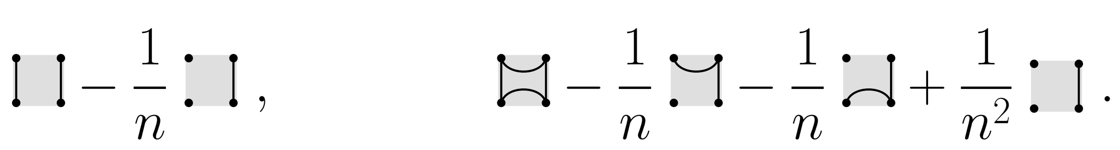
Idempotent $\pi^{\otimes k}_{k+1} := \pi^{\otimes k} \otimes \mathbf{1}$ of $\mathsf{P}_{k+\frac{1}{2}}(n)$ is used to define a half quasipartition algebra:

- $$\mathsf{QP}_{k+\frac{1}{2}}(n) := \pi^{\otimes k}_{k+1}\mathsf{P}_{k+\frac{1}{2}}(n)\pi^{\otimes k}_{k+1},$$
- $$\overline{d} = \pi^{\otimes k}_{k+1} d \pi^{\otimes k}_{k+1}, \quad \text{for } d \in \mathsf{P}_{k+\frac{1}{2}}(n).$$
- ***Proposition*** : $\{\overline{d} \,:\, d \in \mathsf{P}_{k+\frac{1}{2}}(n), d$ *is a set partition with no singletons* $\}$ *is a basis of* $\mathsf{QP}_{k+\frac{1}{2}}(n)$.

Example (Idempotent in $\mathsf{P}_{2+\frac{1}{2}}(n)$/ Identity in $\mathsf{QP}_{2+\frac{1}{2}}(n)$)



Example (Basis of $\mathsf{QP}_{1+\frac{1}{2}}(n)$)



Remark: The algebra $\mathsf{QP}_{1+\frac{1}{2}}(n)$ is not a subalgebra of $\mathsf{QP}_2(n)$.

Intermediate subalgebra of $\mathsf{P}_{k+1}(n)$.

An algebra for which $\mathsf{QP}_{k+\frac{1}{2}}(n)$ is a subalgebra that also projects into $\mathsf{QP}_{k+1}(n)$

- $$\widetilde{\mathsf{QP}}_{k+1}(n) := \pi^{\otimes k}_{k+1}\mathsf{P}_{k+1}(n)\pi^{\otimes k}_{k+1},$$
- $$\tilde{d} = \pi^{\otimes k}_{k+1} d \pi^{\otimes k}_{k+1}, \quad d \in \mathsf{P}_{k+1}(n).$$
- ***Proposition*** : $\{\tilde{d} \,:\, d \in \mathsf{P}_{k+1}(n), d$ *is a set partition with no singletons in* $[k] \cup [\overline{k}]\}$ *is a basis of* $\widetilde{\mathsf{QP}}_{k+1}(n)$.

A chain of algebras

We have the following chain of inclusions and projections:

$$\begin{array}{ccccccccccccc} \mathsf{QP}_0(n) \hookrightarrow & \mathsf{QP}_{\frac{1}{2}}(n) & \subseteq \widetilde{\mathsf{QP}}_1(x) \twoheadrightarrow & \mathsf{QP}_1(n) & \hookrightarrow & \mathsf{QP}_{1+\frac{1}{2}}(n) & \subseteq & \widetilde{\mathsf{QP}}_2(x) & \twoheadrightarrow & \mathsf{QP}_2(n) \hookrightarrow \cdots \\ \cap & \cap & \cap & \cap & & \cap & & \cap & & \cap \\ \mathsf{P}_0(n) \hookrightarrow & \mathsf{P}_{\frac{1}{2}}(n) & \subset \mathsf{P}_1(n) = & \mathsf{P}_1(n) & \hookrightarrow & \mathsf{P}_{1+\frac{1}{2}}(n) & \subset & \mathsf{P}_2(n) & = & \mathsf{P}_2(n) \hookrightarrow \cdots \end{array}$$

The projection is defined by:

$$\widetilde{\mathsf{QP}}_{k+1}(n) \to \mathsf{QP}_{k+1}(n)$$
$$\tilde{d} \mapsto \overline{d} = (\mathbf{1}^{\otimes k+1} - \frac{1}{n}\mathsf{p}_{k+1})\tilde{d}(\mathbf{1}^{\otimes k+1} - \frac{1}{n}\mathsf{p}_{k+1}).$$

Dimensions

- Set partitions of $[k] \cup [\overline{k}]$ with no blocks of size one, counts $\dim(\mathsf{QP}_k(n))$ (see [1]):

$$\dim(\mathsf{QP}_k(n)) = \sum_{j=1}^{2k} (-1)^{j-1} B(2k-j) + 1$$

- $\dim(\mathsf{QP}_{k+\frac{1}{2}}(n)) = B(2k)$ is every other term in the OEIS sequence A000110.
- The dimensions of $\widetilde{\mathsf{QP}}_{k+1}(n)$ are given by the OEIS sequence A207978.

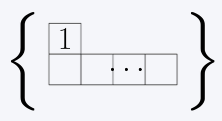
$$\dim(\widetilde{\mathsf{QP}}_{k+1}(n)) = \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} B(2k+2-s)\ .$$

k	0	1	2	3	4	5	6
$\dim(\mathsf{QP}_k(n))$	1	1	4	41	715	17722	580317
$\dim(\mathsf{QP}_{k+\frac{1}{2}}(n))$	1	2	15	203	4140	115975	4213597
$\dim(\widetilde{\mathsf{QP}}_{k+1}(n))$	2	7	67	1080	25287	794545	31858034

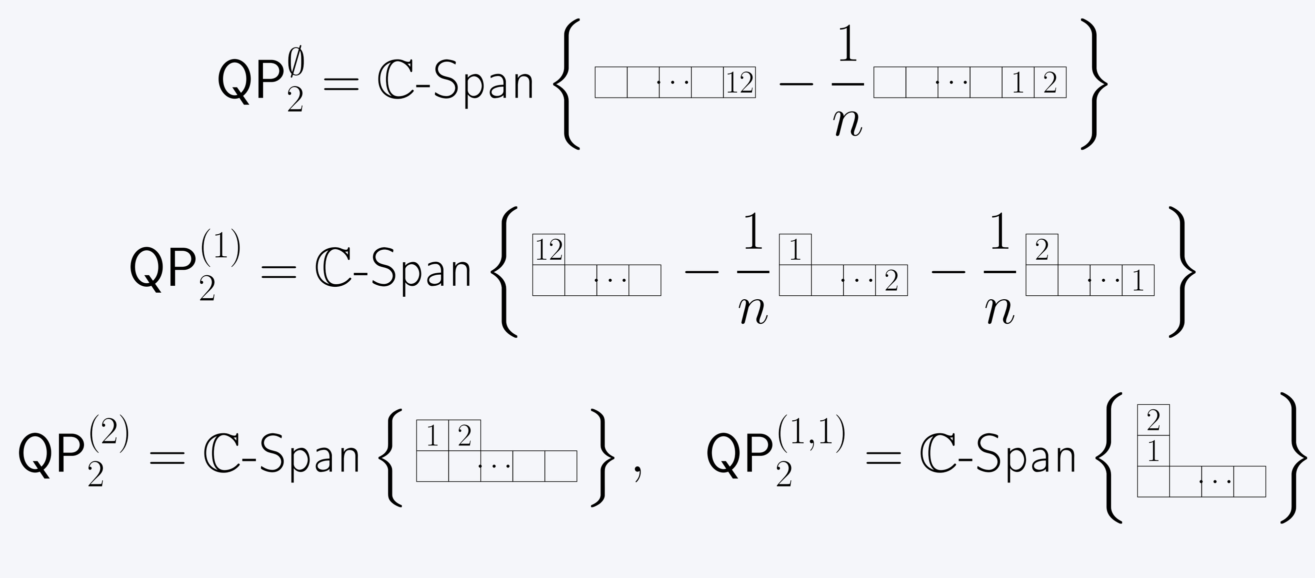
Representations of quasipartition algebras

The $\mathsf{QP}_k(n)$ simple modules are indexed by partitions $|\lambda| \leq k$, and have bases indexed by set valued tableaux of shape $(n-|\lambda|, \lambda)$, no singletons in the first row. For $\mathsf{QP}_1(n)$:

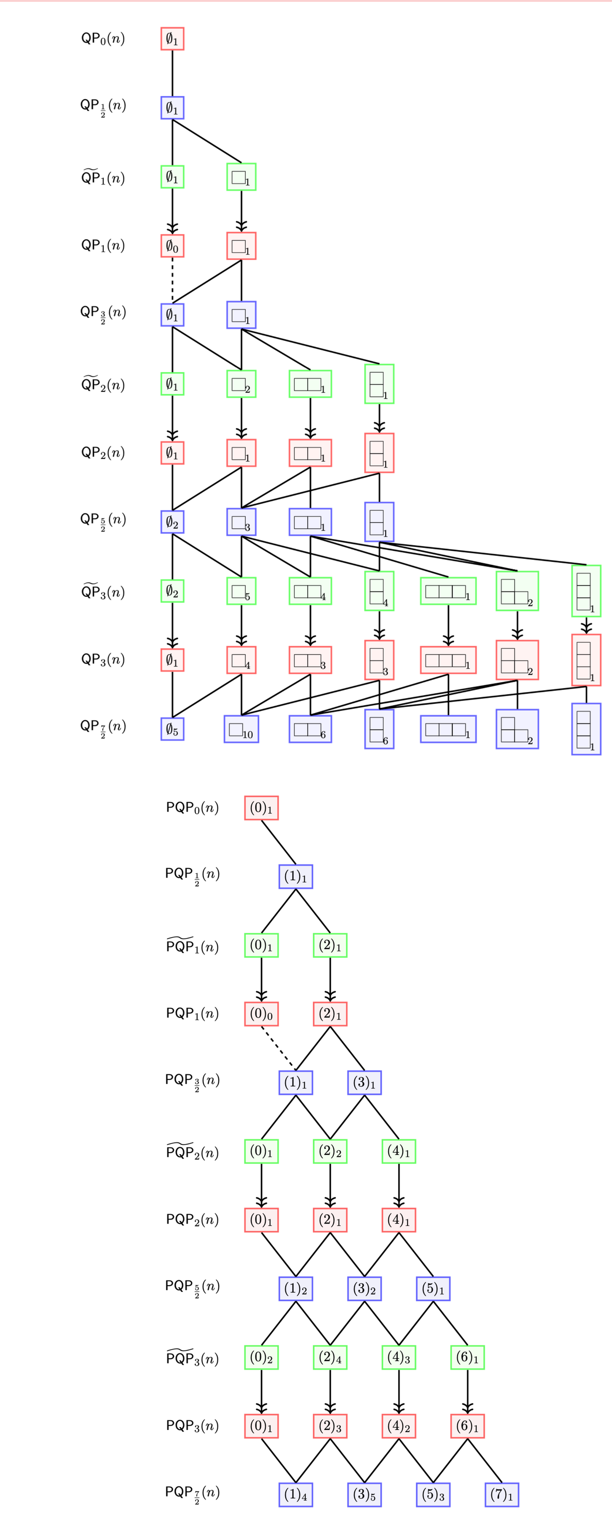
$\mathsf{QP}_1^{\emptyset} = 0 \qquad \mathsf{QP}_1^{(1)} = \mathbb{C}\text{-Span}$



The simple modules for $\mathsf{QP}_2(n)$:



Brattelli like-diagram



Centralizers

Theorem For $n, k \in \mathbb{Z}_{>0}$, if $n \geq 2k$, then

- $\mathsf{QP}_k(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}^{(n-1,1)}_{S_n}\right)^{\otimes k}\right)$,
- $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathrm{End}_{S_{n-1}}\left(\mathrm{Res}^{S_n}_{S_{n-1}}\left(\mathbb{S}^{(n-1,1)}_{S_n}\right)^{\otimes k}\right)$,
- $\widetilde{\mathsf{QP}}_{k+1}(n) \cong \mathrm{End}_{S_n}\left(\left(\mathbb{S}^{(n-1,1)}_{S_n}\right)^{\otimes k} \otimes V_n\right)$.

Corollary $\mathsf{QP}_{k+\frac{1}{2}}(n) \cong \mathsf{P}_k(n-1)$.

Since $\mathrm{Res}^{S_n}_{S_{n-1}} \mathbb{S}^{(n-1,1)}_{S_n} \cong \mathbb{S}^{(n-2,1)}_{S_{n-1}} \oplus \mathbb{S}^{(n-1)}_{S_{n-1}} \cong V_{n-1}$.

References

[1] Zajj Daugherty and Rosa Orellana.
The quasi-partition algebra.
Journal of Algebra, 404:124–151, 2014.

[2] Rosa Orellana, Nancy Wallace, and Mike Zabrocki.
Representations of the quasi-partition algebras.
Journal of Algebra, 655:758–793, 2024.
Special Issue dedicated to the memory of Georgia Benkart.

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