Locally Invariant Vectors in Representations of Symmetric Groups

Amrutha P [1] Amritanshu Prasad [2,3] Velmurugan S [2,3]

[1]Chennai Mathematical Institute [2]The Institute of Mathematical Sciences, Chennai [3]Homi Bhabha National Institute

The Problem: Existence of Locally Invariant Vectors

For which irreducible representations V of a finite group G, and which elements $g \in G$ does there exist a non-zero vector $v \in V$ such that $g \cdot v = v$?

The answer depends on g only through its conjugacy class.

Symmetric Groups

For $G = S_n$, let V_λ be the irreducible representation corresponding to the partition λ of n. Let w_μ denote a permutation with cycle type μ.

Restatement. For which partitions λ and μ does there exist a non-zero vector $v \in V_\lambda$ such that $w_\mu \cdot v = v$?

Main Theorem

The permutation w_μ admits a non-zero invariant vector in V_λ except when

1. $\lambda = (1^n)$, μ is any partition of n for which $w_\mu \notin A_n$,
2. $\lambda = (n-1, 1)$, $\mu = (n)$, $n \geq 2$,
3. $\lambda = (2, 1^{n-2})$, $\mu = (n)$, $n \geq 3$ is odd,
4. $\lambda = (2^2, 1^{n-4})$, $\mu = (n-2, 2)$, $n \geq 5$ is odd,
5. $\lambda = (2, 2)$, $\mu = (3, 1)$,
6. $\lambda = (2^3)$, $\mu = (3, 2, 1)$,
7. $\lambda = (2^4)$, $\mu = (5, 3)$,
8. $\lambda = (4, 4)$, $\mu = (5, 3)$,
9. $\lambda = (2^5)$, $\mu = (5, 3, 2)$.

Alternating Groups

The main theorem was extended to alternating groups in [6]. For every irreducible representation V of A_n, and every $w \in A_n$, there exists a non-zero vector in V that is invariant under w unless one of the following holds:

1. $V = V^{\pm}_{(2,1)}$ and w is a 3-cycle,
2. $V = V^{\pm}_{(2,2)}$ and w has cycle type $(3, 1)$,
3. $V = V_{(4,4)}$ and w has cycle type $(5, 3)$,
4. $V = V_{(n-1,1)}$ and w is an n-cycle, where $n > 3$ is odd.

$V^{\pm}_\lambda$ are the irreducible constitutents of the restriction of V_λ to A_n when λ is self-conjugate.

Cyclic Characters

A *cyclic character* of a finite group G is a character that is induced from a cyclic subgroup of G. The study of cyclic characters goes back to Artin and Brauer, and arose in the context of analytic continuation of *Artin L-functions*.

By Frobenius reciprocity if

$$V_\lambda^{w_\mu} := \{v \in V_\lambda \mid w_\mu v = v\},$$

then

$$\dim V_\lambda^{w_\mu} = \langle \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1, V_\lambda \rangle$$

Our Main Theorem gives necessary and sufficient conditions for the positivity of $[\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1, V_\lambda]$.

Connection to Global Conjugacy Classes

The conjugacy class of an element $g \in G$ is said to be a *global conjugacy class* if the corresponding permutation representation $\mathrm{Ind}_{Z_G(g)}^G 1$ contains every irreducible representation of G.

- Heide and Zalessky [1] conjectured: if every irreducible representation of a finite group occurs in its adjoint representation, then it admits a global conjugacy class. They proved the conjecture for alternating groups and sporadic simple groups.
- Sheila Sundaram [10] characterized all global conjugacy classes for S_n: for $n \neq 4, 8$, $\mu \vdash n$, w_μ lies in a global conjugacy class of S_n if and only if μ has at least two parts and its parts are odd and distinct. Global conjugacy classes exist for $n = 6$ and $n \geq 8$.
- Since $\langle w_\mu \rangle \subset Z_G(w_\mu)$, if the class of w_μ is global, then $V_\lambda^{w_\mu} \neq 0$ for every $\lambda \vdash n$.

The Immersion Poset

Prasad and Ragunathan [8] introduced a partial order on automorphic representations called *immersion*. Adapted to finite dimensional representations of groups:

Say that a representation V of G is said to be *immersed* in a representation W of G if, for every $g \in G$ and every $\lambda \in \mathbb{C}$, the mupltiplicity of λ as an eigenvalue of g in V is less than or equal to the multiplicity of λ as an eigenvalue of g in W.

We write $V \preccurlyeq W$.

Our Main Theorem implies the following result:

Given a partition $\lambda \vdash n$, $V_{(n)} \preccurlyeq V_\lambda$ if and only if λ is not one of

1. (1^n),
2. $(n-1, 1)$ for $n \geq 2$,
3. $(2, 1^{n-2})$ when $n \geq 3$ is odd,
4. $(2^2, 1^{n-4})$, when $n \geq 5$ is odd,
5. $(2, 2)$, (2^3), (2^4), (4^2) and (2^5).

Ingredients of the Proof

1. Let $\chi : \langle w_\mu \rangle \to \mathbb{C}^*$ be a faithful character. Through the work of Klyachko [3], Kraśkiewicz and Weyman [4], Stembridge [9], and Jöllenbeck and Schocker [2] a combinatorial interpretation of $[\mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi^i, V_\lambda]$ as the number of standard Young tableaux of shape λ and *multi-major index* i is obtained. However, the positivity of this multiplicity is difficult to establish from this interpretation.
2. Swanson [11] proved a special case of our main theorem when $\mu = (n)$: $w_{(n)}$ admits a non-zero invariant vector in V_λ except in the following cases:
   1. $\lambda = (n-1, 1)$
   2. $\lambda = (1^n)$ and n is even
   3. $\lambda = (2, 1^{n-2})$ and n is odd.
3. The Littlewood-Richardson rule:
$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$
where $c^\lambda_{\mu\nu}$ is the number of semistandard tableaux of shape λ/μ and content ν whose reverse reading word is a lattice permutation.

Strategy of the Proof

Following a strategy similar to Sundaram, we deduce our main result from Swanson's result by repeated use of the Littlewood-Richardson rule.

Consider the Frobenius characteristic $f_\mu = \mathrm{ch}_n \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} 1$. We wish to prove that

$$f_\mu \geq s_\lambda \qquad (*)$$

for all pairs (μ, λ) barring the exceptions in the Main Theorem.

Since the Young subgroup S_μ contains $\langle w_\mu \rangle$,

$$f_\mu \geq \prod_{i=1}^k f_{(\mu_i)}.$$

But $f_{\mu_i} \geq s_\lambda$ for all partitions $\lambda \vdash \mu_i$ barring the exceptions in Swanson's theorem.

Careful application of the Littlewood-Richardson rule allows us to obtain (*).

Persistence Plays a Role

A $\mu \vdash n$ is called *persistent* if

$$f_\mu \geq s_\lambda \text{ for all } \lambda \vdash n,\ \lambda \neq (1^n).$$

In our proof we first determine which two-part partitions are persistent using Swanson's theorem and the Littlewood-Richardson rule.

We then obtain the Main Theorem with the help of the following lemma:

Lemma. A partition $\mu = (\mu_1, \dots, \mu_k) \vdash n$ with $k \geq 2$ is persistent if the partition $\tilde{\mu}$ obtained by removing a part μ_i from μ is persistent and $n - \mu_i \geq 4$.

Open Questions

1. Classify the global conjugacy classes of A_n.
   We have used our results to construct plethora of new global conjugacy classes for A_n [6].
2. Determine the set of triples (λ, μ, i) such that
$$\langle \mathrm{Ind}_{\langle w_\mu \rangle}^{S_n} \chi^i, V_\lambda \rangle > 0.$$
3. Find an algorithm to construct a standard tableau of shape λ and major index divisible by n for every partition $\lambda \vdash n$ barring the exceptions in Swanson's result.

References

[1] G. Heide and A. E. Zalesski. Passman's problem on adjoint representations. In *Groups, rings and algebras*, volume 420 of *Contemp. Math.*, pages 163–176. Amer. Math. Soc., Providence, RI, 2006.

[2] A. Jöllenbeck and M. Schocker. Cyclic characters of symmetric groups. *J. Algebraic Combin.*, 12(2):155–161, 2000.

[3] A. A. Klyachko. Lie elements in a tensor algebra. *Sibirsk. Mat. Ž.*, 15:1296–1304, 1430, 1974.

[4] W. Kraśkiewicz and J. Weyman. Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.*, (63):265–284, 2001.

[5] Amrutha P, A. Prasad, and Velmurugan S. On the existence of elementwise invariant vectors in representations of symmetric groups, 2023. arXiv:2308.08146.

[6] Amrutha P, Amritanshu Prasad, and Velmurugan S. Cyclic characters of alternating groups, 2023. arXiv:2403.05109.

[7] A. Prasad. *Representation Theory: A Combinatorial Viewpoint*, volume 147 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Delhi, 2015.

[8] D. Prasad and R. Raghunathan. Relations between cusp forms sharing Hecke eigenvalues. *Represent. Theory*, 26:1063–1079, 2022.

[9] J. R. Stembridge. On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math.*, 140(2):353–396, 1989.

[10] S. Sundaram. On conjugacy classes of S_n containing all irreducibles. *Israel J. Math.*, 225(1):321–342, 2018.

[11] J. P. Swanson. On the existence of tableaux with given modular major index. *Algebr. Comb.*, 1(1):3–21, 2018.

[12] N. Yang and A. M. Staroletov. The minimal polynomials of powers of cycles in the ordinary representations of symmetric and alternating groups. *J. Algebra Appl.*, 20(11):Paper No. 2150209, 18, 2021.