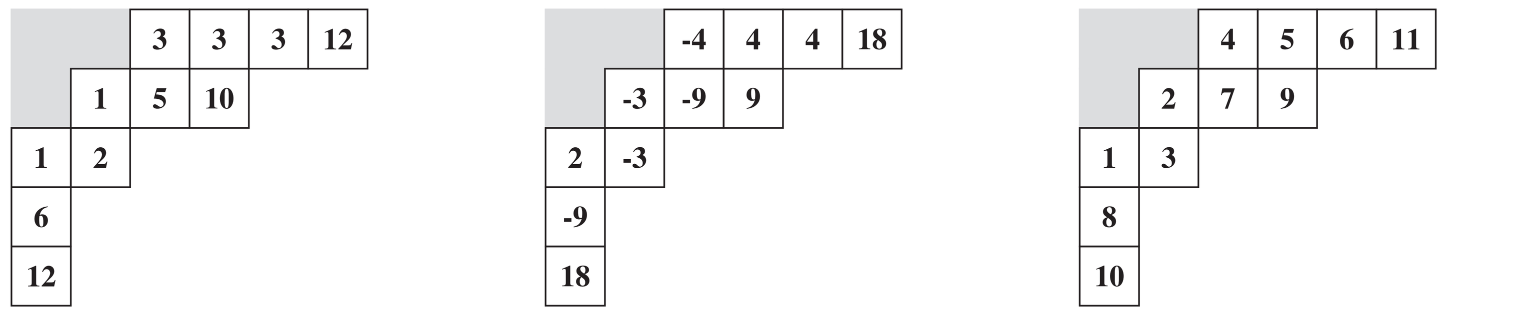
Quasisymmetric expansion of Hall-Littlewood symmetric functions

Darij Grinberg* Ekaterina A. Vassilieva**
Drexel University École Polytechnique

FPSAC, July 22–26, 2024, Ruhr-Universität, Bochum, Germany

Young tableaux and permutation statistics

Let $[n] = \{1, \ldots, n\}$, $\mathbb{P} = \{1, 2, 3, \ldots\}$ and $\mathbb{P}^{\pm} = -\mathbb{P} \cup \mathbb{P}$ be the totally ordered set $-1 < 1 < -2 < 2 < \ldots$. The sets of **semistandard**, **marked** and **standard** Young tableaux of shape λ/μ are denoted $SSYT(\lambda/\mu)$, $SYT^{\pm}(\lambda/\mu)$ and $SYT(\lambda/\mu)$.



The **descent set of a standard Young tableau** T is $\mathrm{Des}(T) = \{1 \leq i \leq n-1 \mid i \text{ is in a strictly higher row than } i+1\}$; its **peak set** is $\mathrm{Peak}(T) = \{2 \leq i \leq n-1 \mid i \in \mathrm{Des}(T) \text{ and } i-1 \notin \mathrm{Des}(T)\}$. For marked tableaux, $\mathrm{neg}(T)$ is the number of negative entries. Denote the descent set and peak set of a permutation π by $\mathrm{Des}(\pi)$ and $\mathrm{Peak}(\pi)$.

Hall-Littlewood S-symmetric functions

Let $X = \{x_1, x_2, x_3, \ldots\}$ and Λ the ring of symmetric functions. Fix a parameter $t \in \mathbb{C}$ and define $q_n(X;t) \in \Lambda$ by $q_0(X;t) = 1$, $q_{\text{negative}}(X;t) = 0$ and

$$q_n(X;t) = (1-t)\sum_i x_i^n \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \qquad \text{for all } n > 0.$$

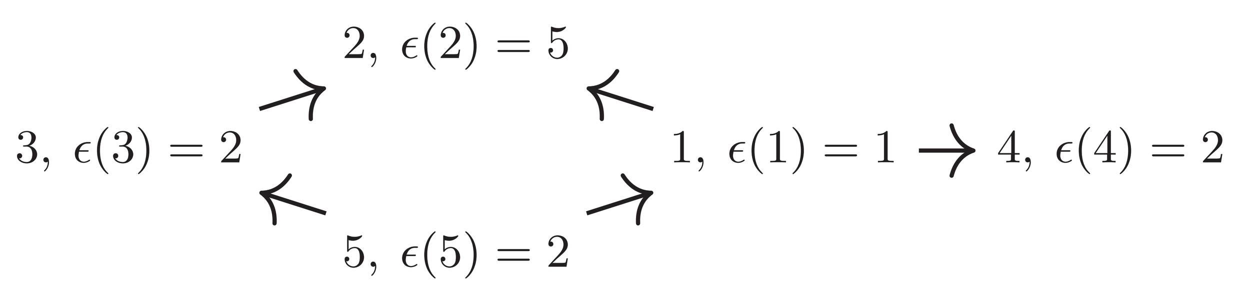
The $(q_n(X;t))_n$ generate a subalgebra Λ_t of Λ which is proper when t is a root of unity.

Def (Hall-Littlewood S-sym. functions). $S_{\lambda/\mu}(X;t) := \det\left(q_{\lambda_i - \mu_j - i + j}(X;t)\right)_{i,j}$. As such, $S_{\lambda/\mu}(X;0) = s_{\lambda/\mu}(X)$ and $S_{\lambda/\mu}(X;-1)$ is a variant of *Schur's Q-function*.

Def (θ_t homomorphism). Define the $\mathbb{C}$-algebra morphism $\theta_t : \Lambda \longrightarrow \Lambda_t$ by $\theta_t(h_n)(X) = q_n(X;t)$. As a consequence, $\theta_t(s_{\lambda/\mu})(X) = S_{\lambda/\mu}(X;t)$.

P-partitions and q-deformed generating functions

A **labelled weighted poset** is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \longrightarrow \mathbb{P}$ is a map (the **weight function**).



The set $\mathcal{L}_{\mathbb{P}^{\pm}}(P)$ of **enriched** P**-partitions** is the set of maps $f : [n] \longrightarrow \mathbb{P}^{\pm}$ that satisfy:

(i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.

(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

Let $X = \{x_1, x_2, x_3, \ldots\}$ and $q \in \mathbb{C}$ (complex numbers). The q**-generating function** for enriched P-partitions on the weighted poset $P = ([n], <_P, \epsilon)$ is

$$\Gamma^{(q)}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}(P)} \prod_{1 \leq i \leq n} q^{[f(i)<0]} x_{|f(i)|}^{\epsilon(i)}.$$

This definition covers the cases of Gessel ($q = 0$) and Stembridge ($q = 1$).

Deformed fundamental quasisymmetric functions

Given a permutation $\pi = \pi_1 \ldots \pi_n$ in the symmetric group $\mathfrak{S}_n$ on $[n]$, we let $P_\pi = ([n], <_\pi, 1^n)$ be the labelled weighted poset on the set $[n]$, where the order relation $<_\pi$ is such that $\pi_i <_\pi \pi_j$ if and only if $i < j$, and where all the weights are equal to 1.

$$\pi_1 \longrightarrow \pi_2 \longrightarrow \cdots\cdots\cdots \longrightarrow \pi_n$$

Def (q-Fundamental quasisymmetric functions). Let π be a permutation in $\mathfrak{S}_n$ and $q \in \mathbb{C}$. Define the *q-fundamental quasisymmetric function* indexed by π as

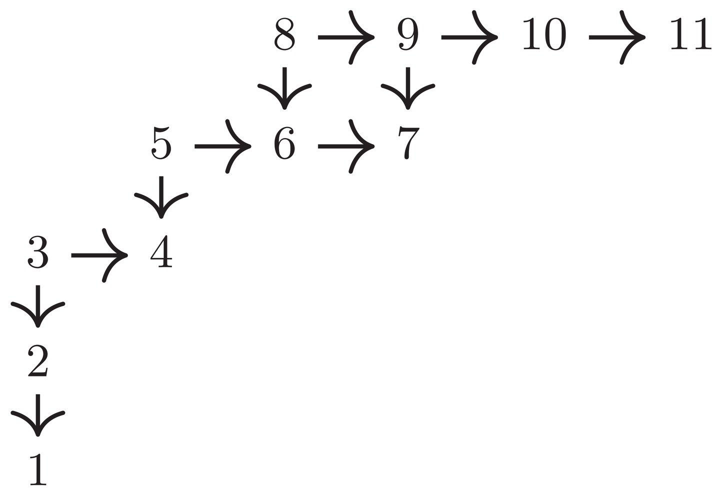
$$L_\pi^{(q)} = \Gamma^{(q)}([n], <_\pi, 1^n) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) \mid i_j = i_{j+1}\}|}(q+1)^{|\{i_1, \ldots, i_n\}|} x_{i_1} \ldots x_{i_n}.$$

$L_\pi^{(q)}$ depends only on $\mathrm{Des}(\pi)$ and may be indexed as $(L_{n,I}^{(q)})_{I \subseteq [n-1], n \geq 0}$. $L_\pi^{(q)}$ interpolates between fundamental ($q = 0$) and peak ($q = 1$) quasisymmetric functions.

Prop. *$(L_{n,I}^{(q)})_{n \geq 0, I \subseteq [n-1]}$ is a basis of QSym if and only if $q \in \mathbb{C}$* ***is not a root of unity*** *and $(L_{n,I}^{q})_{n \geq 0, I \subseteq_p [n-1]}$ is a basis of a proper subalgebra $\mathcal{P}^p$ when $-q$ is a* ***primitive (p+1)-th root of unity****.*

The q-generating functions over skew diagrams

Let $n \in \mathbb{P}$ and λ and μ be two partitions such that λ/μ is a skew shape and $|\lambda| - |\mu| = n$. Label the skew Young diagram of shape λ/μ with the successive integers of $[n]$ from left to right and bottom to top. Define the partial order $<_{\lambda/\mu}$ on $[n]$ as $i <_{\lambda/\mu} j$ if and only if i lies northwest of j and denote the labelled poset $P_{\lambda/\mu} = ([n], <_{\lambda/\mu})$.



Consequently, the enriched $P_{\lambda/\mu}$-partitions are precisely the marked semistandard Young tableaux of shape λ/μ, i.e. $\mathcal{L}_{\mathbb{P}^{\pm}}(P_{\lambda/\mu}) = SSYT^{\pm}(\lambda/\mu)$.

Thm. *Let $n \in \mathbb{P}$ and λ/μ such that $|\lambda| - |\mu| = n$. The q-deformed generating function of $P_{\lambda/\mu}$ is exactly the Hall-Littlewood S-symmetric function with parameter $t = -q$.*

$$S_{\lambda/\mu}(X;-q) = \Gamma^{(q)}([n], <_{\lambda/\mu}).$$

Proof (sketch). Let $(i\,R\,j) \iff (i \leq j \text{ but not } i = j \in -\mathbb{P})$. Let $X^{\pm} = \{x_{-1}, x_1, x_{-2}, x_2, \ldots\}$ and

$$H_n(X^{\pm}) = \sum_{\substack{(i_1, \ldots, i_n) \in (\mathbb{P}^{\pm})^n; \\ i_1\, R \cdots R\, i_n}} x_{i_1} \cdots x_{i_n}, \qquad \Gamma^{\pm}([n], <_P)(X^{\pm}) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{1 \leq i \leq n} x_{f(i)}.$$

Use [2] to show that $\Gamma^{\pm}([n], <_{\lambda/\mu}) = \det\left(H_{\lambda_i - \mu_j - i + j}\right)_{i,j \in [k]}$ and prove that the homomorphism $\varpi(x_i) = q^{[i<0]} x_{|i|}$ for $x_i \in X^{\pm}$ is such that $\varpi(H_n(X^{\pm})) = q_n(X;-q)$. □

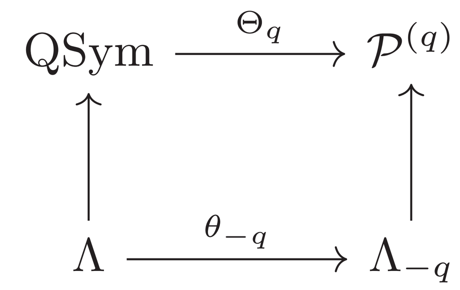
Relating Hall-Littlewood and q-fundamentals

Thm. *Let λ/μ be a skew shape. The Hall-Littlewood S-symmetric function with parameter $t = -q$ is related to q-fundamental quasisymmetric functions through*

$$S_{\lambda/\mu}(X;-q) = \sum_{T \in SYT(\lambda/\mu)} L_{\mathrm{Des}(T)}^{(q)}(X).$$

Proof (sketch). Show that $S_{\lambda/\mu}(X;-q) = \sum_{T \in SSYT^{\pm}(\lambda/\mu)} q^{\mathrm{neg}(T)} X^{|T|}$ and group the summands according to the standardisation of their marked tableau. □

Thm. *There is a ring homomorphism Θ_q: $\mathrm{QSym} \longrightarrow \mathcal{P}^{(q)}$ such that for any positive integer n and any subset $I \subseteq [n-1]$, $\Theta_q\left(L_{n,I}^{(0)}\right) = L_{n,I}^{(q)}$. The restriction of Θ_q to Λ is θ_{-q} and the following ring map diagram is commutative:*



Cauchy-like formula for Hall-Littlewood functions

Denote $Y = \{y_1, y_2, \ldots,\}$ and $XY = \{x_i y_j\}_{i,j}$. Let π be a permutation. Extend $\Gamma^{(q)}$ to the alphabet XY by considering P_π-partitions $(f,g) : i \mapsto (f(i), g(i))$ with value in $\mathbb{P} \times \mathbb{P}^{\pm}$. Say that a pair $(i,j) \in \mathbb{P} \times \mathbb{P}^{\pm}$ is negative if and only if j is negative.

$$\Gamma^{(q)}(P_\pi)(XY) = \sum_{(f,g) \in \mathcal{L}_{\mathbb{P} \times \mathbb{P}^{\pm}}([n], <_\pi)} \prod_{1 \leq i \leq n} q^{[g(i)<0]} x_{f(i)} y_{|g(i)|}.$$

The q-fundamental indexed by π on the product of indeterminate XY satisfies

$$L_\pi^{(q)}(XY) = \Gamma^{(q)}(P_\pi)(XY) = \sum_{\sigma \circ \tau = \pi} L_\sigma^{(0)}(X) L_\tau^{(q)}(Y).$$

Prop. *Our results directly imply the following Cauchy-like formula (see [4])*

$$q_n(XY;t) = \sum_{\lambda \vdash n} s_\lambda(X) S_\lambda(Y;t).$$

References

[1] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, 1984

[2] I. Gessel and X. Viennot. Determinants, Paths, and Plane Partitions; 1989.

[3] J. Stembridge, Enriched P-partitions, 1997

[4] I. Macdonald. Symmetric functions and Hall polynomials. 1999.

[5] D. Grinberg and E. Vassilieva, Weighted posets and the enriched monomial basis of QSym, 2021

[6] D. Grinberg and E. Vassilieva, A q-deformation of enriched P-partitions, 2022

[7] D. Grinberg and E. Vassilieva, The algebra of extended peaks, 2023

* darijgrinberg@gmail.com ** katya@lix.polytechnique.fr