Charge formulas for Macdonald polynomials at $t = 0$ from multiline queues and diagrams

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Abstract

Multiline queues are combinatorial objects coming from probability theory that give formulas for the q-Whittaker specialization $P_\lambda(X;q,0)$ of the Macdonald polynomials. We define a charge statistic and an RSK-esque procedure on multiline queues that naturally recovers the Schur expansion of $P_\lambda(X;q,0)$. We extend these results to *generalized multiline queues*, which are in bijection with binary matrices, and obtain a new family of formulas for $P_\lambda(X;q,0)$ in terms of these objects.

Multiline diagrams are the plethystic analogues of multiline queues that were recently found to give a formula for the modified Hall–Littlewood polynomials $\widetilde{H}_\lambda(X;q,0)$. We obtain formulas for the latter through a cocharge statistic and an RSK-esque procedure on multiline diagrams.

Multiline queues and Macdonald polynomials

Fix a **partition** λ and $n \in \mathbb{N}$. A **multiline queue** of *shape* (λ, n) is a $\lambda_1 \times n$ **binary matrix with row content** λ', representing a configuration of particles (1) and vacancies (0). The set of such objects is denoted $\mathrm{MLQ}(\lambda, n)$.

A t**-multiline queue** of shape (λ, n) (denoted $t\text{-MLQ}(\lambda,n)$) is a configuration of particles together with a **pairing procedure** in which particles in consecutive rows are cylindrically paired *weakly to the right*. In [1], it is shown that the symmetric Macdonald polynomial is given by

$$P_\lambda(x_1, x_2 \ldots, x_n, q, t) = \sum_{M \in t\text{-MLQ}(\lambda,n)} \mathrm{wt}_{X,q,t}(M)$$

where $\mathrm{wt}_{X,q,t}$ is a weight function in parameters x_i, q, and t, with x_i's recording the **column content**, q recording the **wrapping pairings**, and t recording the **"skipping of balls"** in the pairing. Setting $t = 0$ restricts $t\text{-MLQ}(\lambda,n)$ to the set $\mathrm{MLQ}(\lambda,n)$, with pairings dictated by the (deterministic) Ferrari–Martin pairing algorithm [2] in which each particle is paired to the first available one, and particles are **labelled** according to the pairings.

For $M \in \mathrm{MLQ}(\lambda,n)$, let $m_{r,\ell}$ be the number of **wrapping pairings** of particles labelled ℓ from row r to row $r-1$. Then the **major index** of M is

$$\mathrm{maj}(M) = \sum_{r,\ell} m_{r,\ell}(\ell - r + 1).$$

Denote the **non-wrapping multiline queues** by $\mathrm{MLQ}_0(\lambda,n) := \{M \in \mathrm{MLQ}(\lambda,n) : \mathrm{maj}(M) = 0\}$.

Collapsing on multiline queues

For a permutation $\sigma \in \mathfrak{S}_n$, its **charge** is $\mathrm{charge}(\sigma) = \mathrm{maj}(\mathrm{rev}(\sigma^{-1})) = \sum_{i \notin \mathrm{Des}(\sigma)}(n-i)$.

For w with partition content μ, its **charge** is the sum over the charges of its **standard subwords**. The **cocharge** of such a word is $\mathrm{cocharge}(w) = n(\mu) - \mathrm{charge}(w)$.

Example: For $w = 3\,4\,2\,2\,3\,2\,2\,1\,1\,5\,1\,1\,3\,4$ the charge is

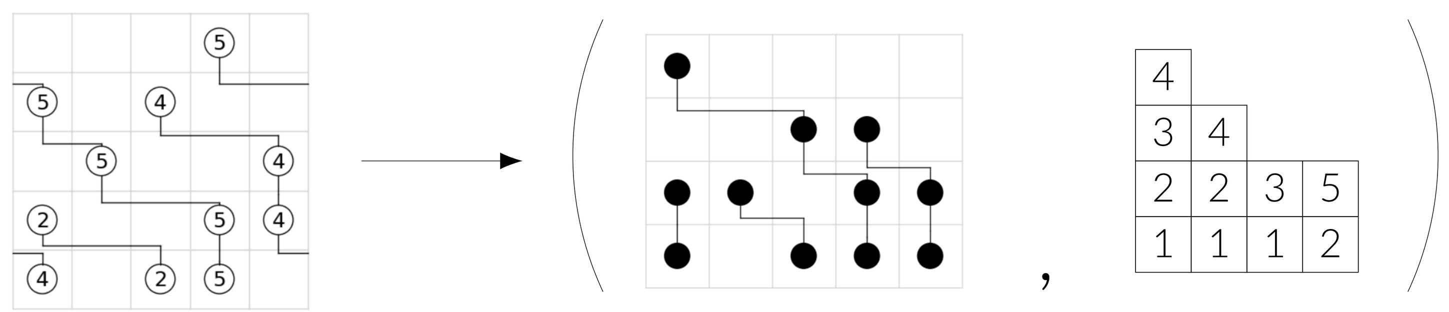
$$\mathrm{charge}(w) = \mathrm{charge}(4\,3\,2\,5\,1) + \mathrm{charge}(3\,2\,1\,4) + \mathrm{charge}(2\,1\,3) + \mathrm{charge}(2\,1) = 1+1+1+0 = 3$$

Lemma: For a multiline queue M, let $\mathrm{cw}(M)$ be the word recording the row labels from top to bottom of the columns of M read from left to right. Then $\mathrm{maj}(M) = \mathrm{charge}(\mathrm{cw}(M))$.

Theorem: The **collapsing map** is a **weight-preserving** bijection:

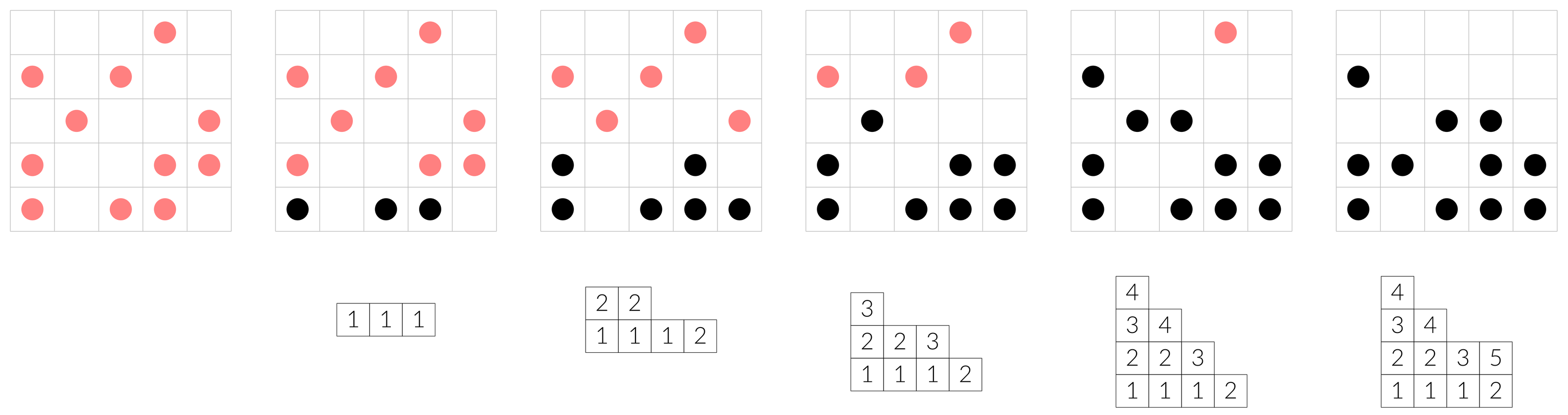
$$\rho : \mathrm{MLQ}(\lambda,n) \longrightarrow \bigcup_{\mu \le \lambda} \mathrm{MLQ}_0(\mu,n) \times \mathrm{SSYT}(\mu', \lambda')$$

with $x^M q^{\mathrm{maj}(N)} = x^N q^{\mathrm{charge}(Q)}$ for $(N,Q) := \rho(M)$.



$\lambda = (5,4,2)$ $\mathrm{maj}(M) = 4$ | $\mu = (4,3,2,2)$ $\mathrm{maj}(N) = 0$ | $\mathrm{charge}(Q) = 4$

The image of the collapsing map $\rho(M)$ is obtained by applying row operators that act as **crystal operators** on $\mathrm{cw}(M)$.



Corollary: The expansion of the q-**Whittaker polynomials** in terms of multiline queues is

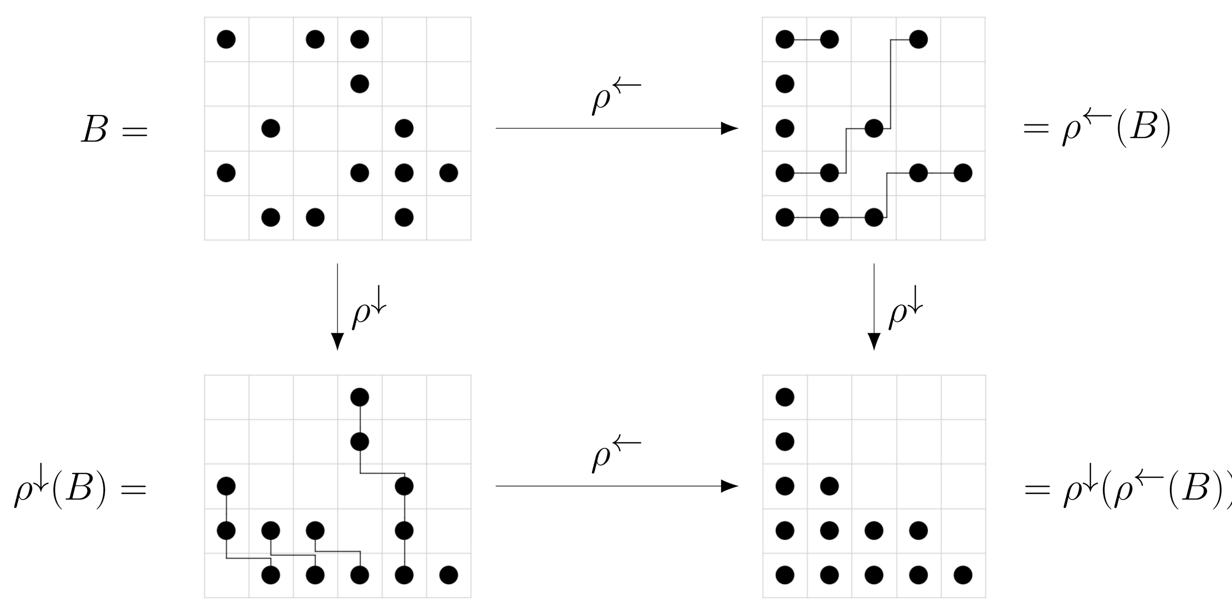
$$P_\lambda(x_1,\ldots,x_n;q,0) := \sum_\mu K_{\mu'\lambda'}(q) s_\mu(x_1,\ldots,x_n) = \sum_{M \in \mathrm{MLQ}(\lambda,n)} q^{\mathrm{charge}(\mathrm{cw}(M))} x^M.$$

Moreover,

$$K_{\lambda\mu}(q) = \sum_{\substack{M \in \mathrm{MLQ}(\mu,\lambda') \\ \rho_N(M) = M(\lambda')}} q^{\mathrm{maj}(M)} = \sum_{N \in \mathrm{MLQ}_0(\lambda,\mu')} q^{\mathrm{maj}(\mathrm{rot}(N))}.$$

Multiline queue RSK

Theorem: $\mathrm{mRSK} : \mathcal{M}_2 \to \bigcup_\lambda \mathrm{MLQ}_0(\lambda) \times \mathrm{MLQ}_0(\lambda')$ given by $\mathrm{mRSK}(B) = (\rho^\downarrow(B), \rho^\leftarrow(B))$ is a bijection.



Corollary: (Dual Cauchy identity) $\displaystyle\prod_{i,j\ge 1}(1 + x_i y_j) = \sum_\lambda s_\lambda(X) s_{\lambda'}(Y)$

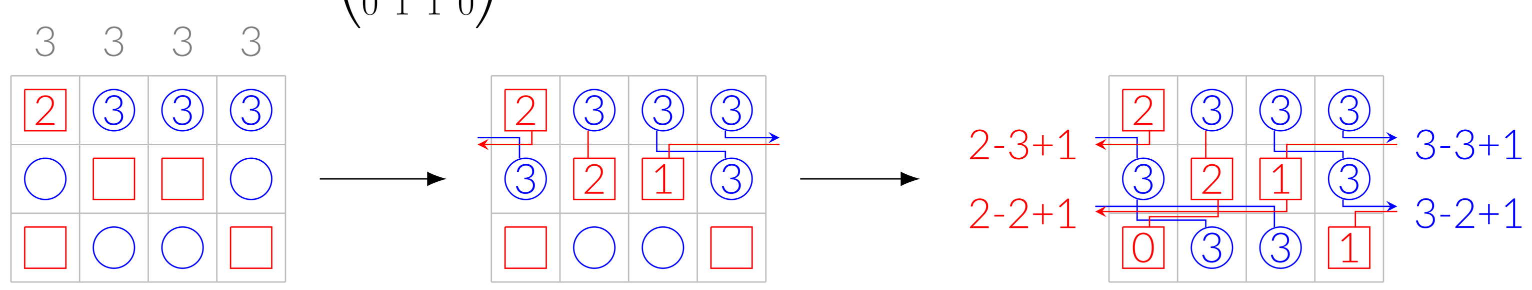
Generalized Multiline Queues

The Ferrari-Martin algorithm can be extended to **arbitrary binary matrices** [3] by treating vacancies as **antiparticles** which pair weakly to the left. The (anti)particles are labelled in two phases of the FM procedure. After the pairing process, we denote the set of binary matrices with row content α and n columns by $\mathrm{GMLQ}(\alpha, n)$.

For $1 \le r, \ell \le L$, let $m_{r,\ell}$ (*resp.* $a_{r,\ell}$) be the number of particles (*resp.* anti-particles) of type ℓ that wrap when pairing to the right (*resp.* left) from row r to row $r-1$. Define

$$\mathrm{maj}_G(B) = \sum_{1 \le r,\ell \le L} m_{r,\ell}(\ell - r + 1) - a_{r,\ell}(\ell - r + 1).$$

Example: For $B = \begin{pmatrix} 0&1&1&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix} \in \mathrm{GMLQ}((2,2,3),4)$ we have $\mathrm{maj}_G(B) = 1 + 2 - 0 - 1 = 2$.



Theorem: Let λ be a partition, $n \in \mathbb{N}$, and let α be a composition with $\alpha^+ = \lambda'$. Then

$$P_\lambda(x_1,\ldots,x_n;q,0) = \sum_{B \in \mathrm{GMLQ}(\alpha,n)} q^{\mathrm{maj}_G(B)} x^B.$$

Multiline diagrams and cocharge

Multiline diagrams (denoted $\mathrm{MLD}(\lambda,n)$) are analogues to multiline queues for **nonnegative integer matrices** where the pairing process is done **strictly to the left.**

The **major index** of a multiline diagram D, denoted $\widetilde{\mathrm{maj}}(D)$, is given by the **non-wrapping pairings**:

$$\widetilde{\mathrm{maj}}(D) = n(\lambda) - \sum_{r,\ell} m_{r,\ell}(r - \ell + 1), \qquad n(\lambda) = \sum_i \binom{\lambda_i}{2}$$

Denote **non-wrapping multiline diagrams** by $\mathrm{MLD}_0(\lambda,n) := \{D \in \mathrm{MLQ}(\lambda,n) : \widetilde{\mathrm{maj}}(D) = n(\lambda)\}$.

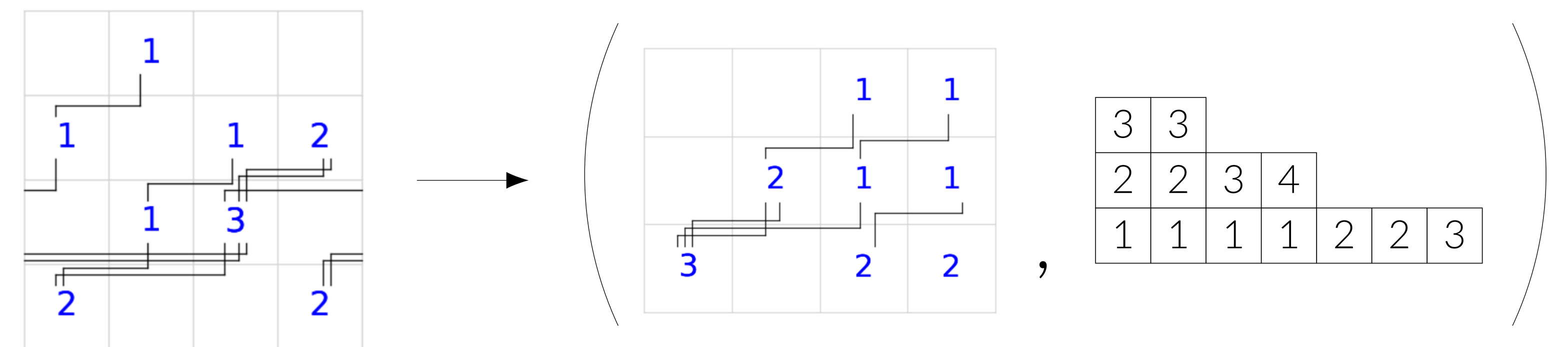
Lemma: For a multiline diagram D we have $\widetilde{\mathrm{maj}}(D) = \mathrm{cocharge}(\mathrm{rev}(\mathrm{cw}(D)))$.

Collapsing on multiline diagrams

Theorem: The collapsing map is a is a **weight-preserving** bijection

$$\widetilde{\rho} : \mathrm{MLD}(\lambda,n) \longrightarrow \bigcup_{\mu \le \lambda} \mathrm{MLD}_0(\mu,n) \times \mathrm{SSYT}(\mu',\lambda')$$

with $x^D q^{\widetilde{\mathrm{maj}}(D)} = x^{\widetilde{N}} q^{\mathrm{cocharge}(\widetilde{Q})}$ for $(\widetilde{N}, \widetilde{Q}) := \widetilde{\rho}(D)$.



$\lambda = (4,3,3,2)$ $\widetilde{\mathrm{maj}}(D) = 9$ | $\mu = (3,3,2,2,1,1,1)$ $\widetilde{\mathrm{maj}}(\widetilde{N}) = n(\mu) = 8$ | $\mathrm{cocharge}(\widetilde{Q}) = 9$

Corollary: The expansion of the **modified Hall–Littlewood polynomials** in terms of multiline diagrams is

$$\widetilde{H}_\lambda(x_1,\ldots,x_n;q,0) := \sum_\mu \widetilde{K}_{\lambda\mu}(q) s_\mu(x_1,\ldots,x_n) = \sum_{D \in \mathrm{MLD}(\lambda,n)} q^{\widetilde{\mathrm{maj}}(D)} x^D.$$

Moreover,

$$\widetilde{K}_{\lambda\mu}(q) = \sum_{\substack{D \in \mathrm{MLD}(\mu',\lambda) \\ \widetilde{\rho}_N(D) = D(\lambda')}} q^{\widetilde{\mathrm{maj}}(D)} = \sum_{D \in \mathrm{MLD}(\lambda',\mu)} q^{\widetilde{\mathrm{maj}}(\mathrm{rot}(D))}$$

Multiline diagram RSK

Theorem: The map $\mathrm{dRSK} : \mathcal{M} \to \bigcup_\lambda \mathrm{MLD}_0(\lambda) \times \mathrm{MLD}_0(\lambda)$ given by $\mathrm{dRSK}(T) = (\widetilde{\rho}^\downarrow(T), \widetilde{\rho}^\leftarrow(T))$ is a bijection.

Corollary: (Cauchy identity) $\displaystyle\prod_{i,j\ge 1}\frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(X) s_\lambda(Y)$

References

[1] S. Corteel, O. Mandelshtam, and L. K. Williams, "From multiline queues to Macdonald polynomials via the exclusion process," *Am. J. Math.*, vol. 144, pp. 395 – 436, 2018.

[2] P. A. Ferrari and J. B. Martin, "Stationary distributions of multi-type totally asymmetric exclusion processes," *Ann. Probab.*, vol. 35, pp. 807–832, 2007.

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[4] A. Ayyer, O. Mandelshtam, and J. B. Martin, "Modified Macdonald polynomials and the multispecies zero range process: II," 2022. *Math. Z.*, to appear.

[5] O. Mandelshtam and J. Valencia-Porras, "Macdonald polynomials at $t = 0$ through (generalized) multiline queues," 2024.