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Vines and MAT-labeled graphs

Tan Nhat Tran (Leibniz University Hannover, Germany, tan.tran@math.uni-hannover.de)
joint work with Hung Manh Tran (Singapore) and Shuhei Tsujie (Hokkaido)

Introduction

Sommers-Tymoczko 2006, Abe-Barakat-Cuntz-Hoge-Terao 2016

Let Φ be an irreducible root system and $\mathcal{I} \subseteq \Phi^+$ an **ideal** (downward-closed subset) of the root poset. Then the ideal subarrangement

$$\mathcal{A}_\mathcal{I} = \{\alpha^\perp \mid \alpha \in \mathcal{I}\} \text{ is MAT-free} \implies \mathcal{A}_\mathcal{I} \text{ is free.}$$

Definition

Let $G = (V_G, E_G)$ be an undirected, simple graph with $|V_G| = \ell$. The **graphic arrangement** $\mathcal{A}_G$ in $\mathbb{K}^\ell$ is defined by

$$\mathcal{A}_G := \{x_i - x_j = 0 \mid \{v_i, v_j\} \in E_G\}.$$

Stanley 1972, Edelman-Reiner 1994

$\mathcal{A}_G$ is free $\iff$ G is **chordal**, i.e. G is C_n-free for $n \geq 4$, where C_n is the **cycle graph**.

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$\mathcal{A}_G$ is MAT-free $\iff$ G is **strongly chordal**, i.e. G is chordal and S_n-free for $n \geq 3$, where S_n is the **sun graph**.

Question (Cuntz-Mücksch 2020)

Can we characterize the MAT-freeness by a poset structure generalizing the classical root poset?

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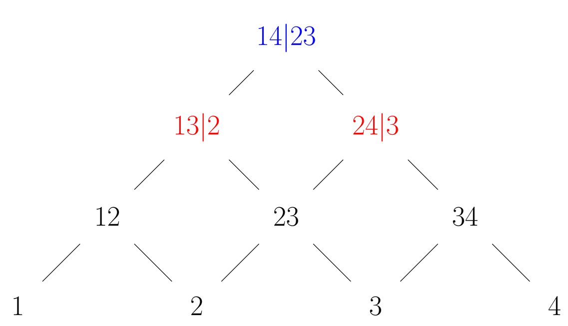
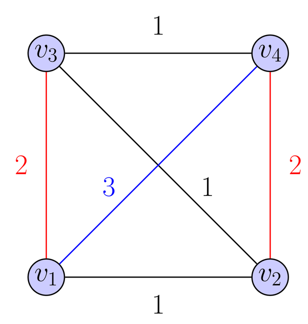
YES for graphic arrangements! The MAT-freeness is completely characterized by **locally regular vines** from **probability theory**.

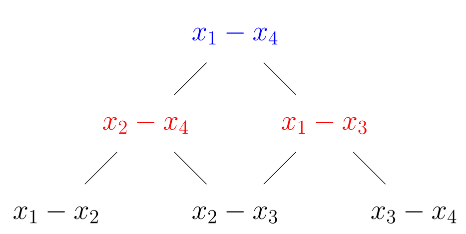
Examples



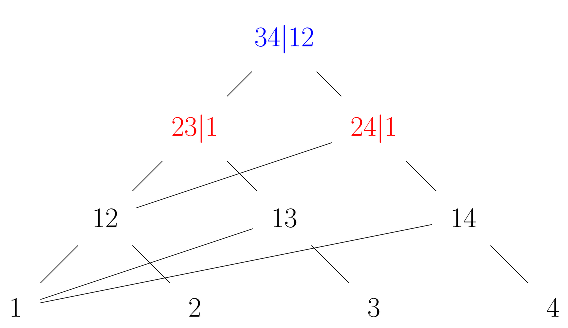
D-vine = type A root poset



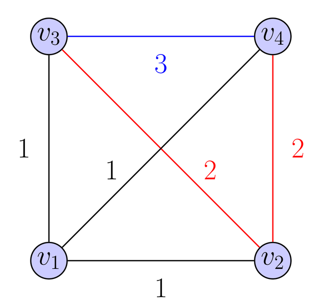
MAT-labeled graph



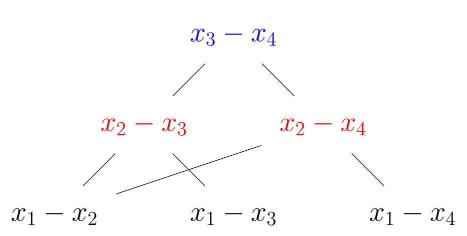
MAT-partition



C-vine



MAT-labeled graph



MAT-partition

Free arrangements (Saito 1980, Terao 1980)

Let $\mathbb{K}$ be a field, and $S = \mathbb{K}[x_1, \ldots, x_\ell]$. Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^\ell$, and $Q = Q(\mathcal{A})$ the defining polynomial of $\mathcal{A}$. Let $\mathrm{Der}(S) = \bigoplus_{i=1}^\ell S\partial_{x_i}$. The **module of $\mathcal{A}$-derivations** is defined by

$$D(\mathcal{A}) := \{\varphi \in \mathrm{Der}(S) \mid \varphi(Q) \in QS\}.$$

The arrangement $\mathcal{A}$ is called **free** if $D(\mathcal{A})$ is a free S-module.

MAT-free arrangements (ABCHT 2016, CM 2020)

Let $\mathcal{A} \neq \varnothing$. A partition $\pi = (\pi_1, \ldots, \pi_n)$ of $\mathcal{A}$ is called an **MAT-partition** if the following conditions hold for every $1 \leq k \leq n$:

(1) The hyperplanes in π_k are linearly independent.
(2) $\nexists H' \in \mathcal{A}_{k-1}$ such that $\bigcap_{H \in \pi_k} H \subseteq H'$, where $\mathcal{A}_{k-1} := \pi_1 \sqcup \cdots \sqcup \pi_{k-1}$ and $\mathcal{A}_0 := \varnothing$.
(3) For each $H \in \pi_k$, $|\mathcal{A}_{k-1}| - |(\mathcal{A}_{k-1} \cup \{H\})^H| = k - 1$.

An arrangement is called **MAT-free** if it is empty or admits an MAT-partition.

MAT-labelings (TT 2023)

Given a graph G, a labeling $\lambda \colon E_G \longrightarrow \mathbb{Z}_{>0}$ is an **MAT-labeling** if the following conditions hold for every $k \geq 1$:

(1) An edge of label $\leq k$ doesn't form a cycle with edges of label k.
(2) Every edge of label k forms exactly $k-1$ triangles with edges of label $< k$.

(G, λ) is called an **MAT-labeled graph** if λ is an MAT-labeling.

LR-vines (Joe 1994, Cooke 1997, Bedford-Cooke 2001, TTT 2024+)

A **locally regular vine** (or **LR-vine**) $\mathcal{P}$ is a finite graded poset that satisfies:

(1) Every non-minimal node covers exactly two other nodes, and any two distinct nodes of the same rank are covered by at most one node.
(2) For each $1 \leq i \leq \mathrm{rk}(\mathcal{P})$, the graph $F_i = (\mathcal{P}_i, \mathcal{P}_{i+1}^{\leq})$ is a **forest** where $\mathcal{P}_{i+1}^{\leq}$ denotes the sets of pairs in $\mathcal{P}_i$ covered by $\mathcal{P}_{i+1}$.
(3) **Proximity**: For any distinct non-minimal nodes a, b of the same rank, if a and b are covered by a common node, then a and b cover a common node.

Main result (TTT 2023+)

There exists an **equivalence** between the category of **MAT-labeled graphs** and the category of **LR-vines**.

Main references

[1] T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao. The freeness of ideal subarrangements of Weyl arrangements. *J. Eur. Math. Soc. (JEMS)*, 18(6):1339–1348, 2016.
[2] H. M. Tran, T. N. Tran, and S. Tsujie. Vines and MAT-labeled graphs. arXiv:2311.17793.
[3] T. N. Tran and S. Tsujie. MAT-free graphic arrangements and a characterization of strongly chordal graphs by edge-labeling. *Algebr. Comb.*, 6(6):1447–1467, 2023.