A Consistent Sandpile Torsor Algorithm for Regular Matroids

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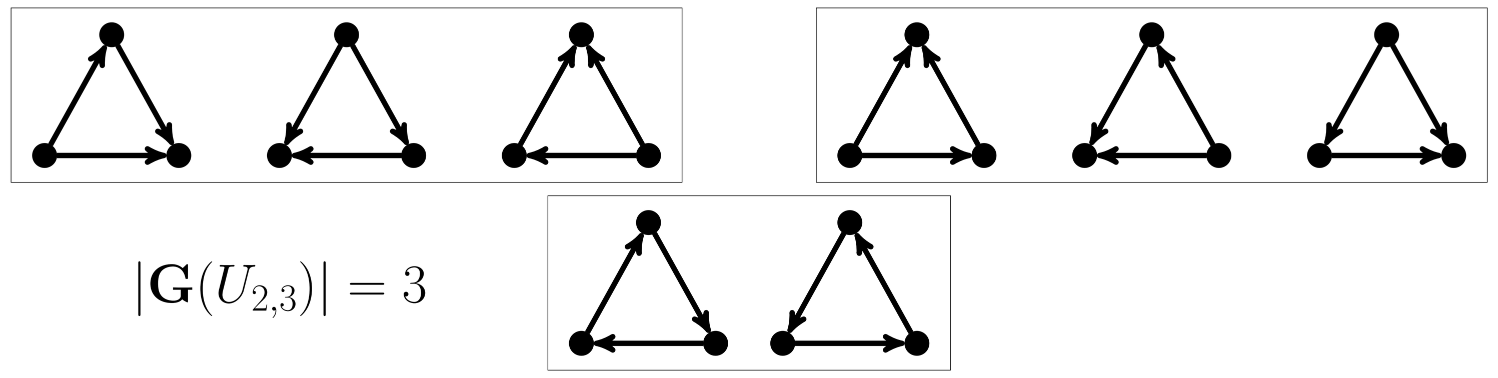
Introduction

- Every regular matroid is associated with a *sandpile group*, which acts simply transitively on the set of bases in various ways.
- A family of group actions is *consistent* if it respects **deletion-contraction**.
- We prove the group action constructed by Backman–Santos–Yuen and Ding is consistent, generalizing the result by Ganguly–McDonough for planar graphs.

Setting

- M: a regular matroid on E; $\mathbf{B}(M)$: bases of M.
- Sandpile group $S(M)$: $\frac{\mathbb{Z}^E}{\Lambda(M)\oplus\Lambda^*(M)}$, where $\Lambda(M)$ (resp. $\Lambda^*(M)$) is the sublattice generated by all signed circuits (resp. cocircuits) of M.
- Circuit-cocircuit reversal system $\mathbf{G}(M)$: two orientations of M are equivalent if one can be obtained from the other by reversing directed circuits/cocircuits; $\mathbf{G}(M)$ consists of the equivalence classes of orientations.



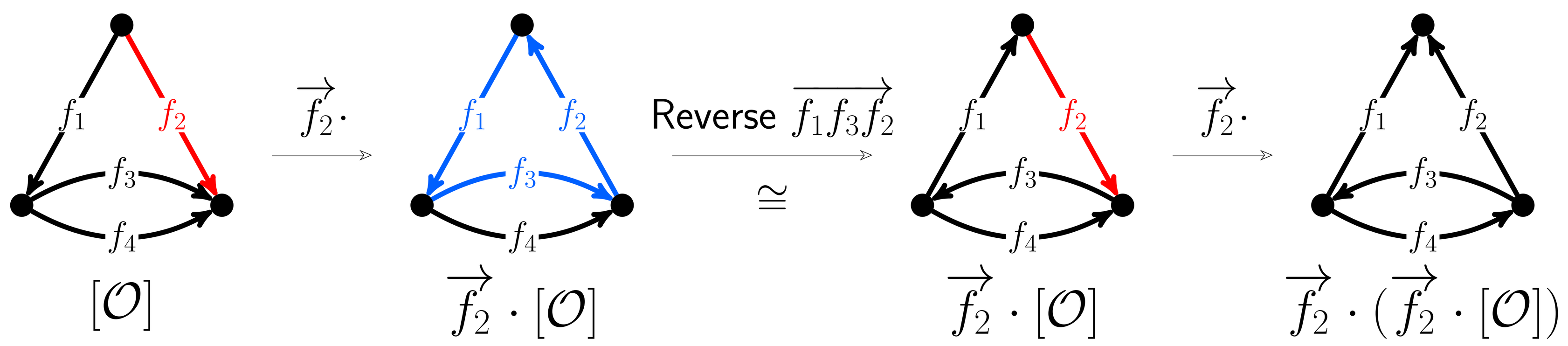
$|\mathbf{G}(U_{2,3})| = 3$

Canonical Action $S(M) \circlearrowright \mathbf{G}(M)$

An oriented element $\overrightarrow{f} \in \mathbb{Z}^E$ acts on a reversal class $[\mathcal{O}]$:

- If f is oriented as $\overrightarrow{f}$ in $\mathcal{O}$, pick a directed circuit/cocircuit of $\mathcal{O}$ containing f and reverse it. (Such a circuit or cocircuit always exists.)
- Reverse the orientation of f.

Extend linearly to obtain $\mathbb{Z}^E \circlearrowright \mathbf{G}(M)$ which descends to $S(M) \circlearrowright \mathbf{G}(M)$.



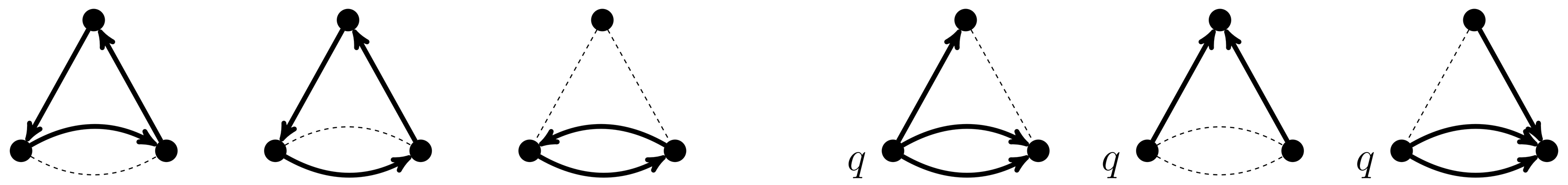
Proposition (Backman–Baker–Yuen 2019)

The canonical action is well-defined and simply transitive.

BBY Bijection $\mathbf{B}(M) \longleftrightarrow \mathbf{G}(M)$

A pair of *signatures* (σ, σ^*) picks an orientation for every circuit and cocircuit of M; it is *triangulating* if it satisfies certain non-overlapping condition.

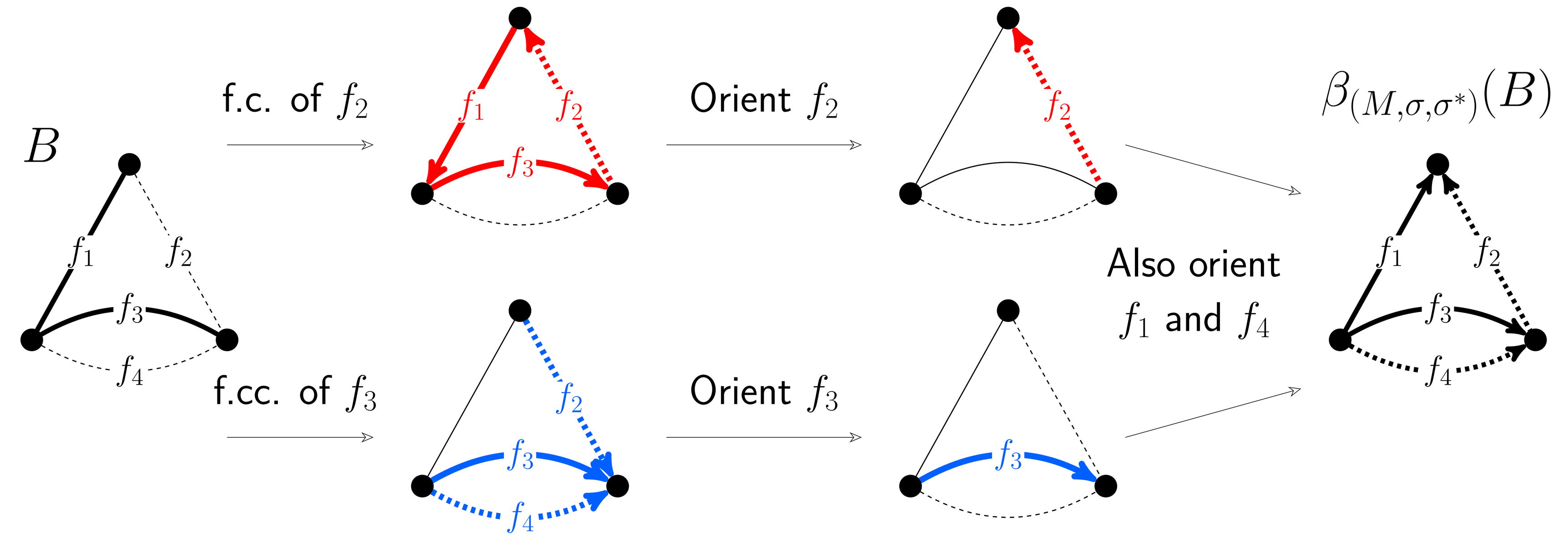
Example: for a plane graph, orient each circuit counterclockwise; orient each cocircuit away from a fixed vertex q.



Fact: A triangulating signature σ (resp. σ^*)
⇔ a triangulation of the Lawrence polytope of M (resp. M^*)
⇔ a generic single-element lifting (resp. extension) of M.

Given a pair (σ, σ^*) of signatures and a basis $B \in \mathbf{B}(M)$, we can produce an orientation $\beta_{(M,\sigma,\sigma^*)}(B)$ (thus a class in $\mathbf{G}(M)$):

- For every $f \in B$, orient f according to $\sigma^*(D)$, where D is the fundamental cocircuit of f w.r.t. B.
- For every $f \notin B$, orient f according to $\sigma(C)$, where C is the fundamental circuit of f w.r.t. B.



Proposition (Backman–Santos–Yuen 2020, Ding 2023+)

For a pair (σ, σ^*) of triangulating signatures, the BBY map $\beta_{(M,\sigma,\sigma^*)}$ is a bijection between $\mathbf{B}(M)$ and $\mathbf{G}(M)$.

BBY Action $S(M) \circlearrowright \mathbf{B}(M)$

For a regular matroid M equipped with a pair of triangulating signatures (σ, σ^*), an element $\gamma \in S(M)$, and a basis B_1, we define the BBY action

$$\gamma \cdot B_1 := B_2 \iff \gamma \cdot \beta_{(M,\sigma,\sigma^*)}(B_1) = \beta_{(M,\sigma,\sigma^*)}(B_2).$$

That is, we compose the aforementioned constructions:

$$S(M) \circlearrowright \mathbf{G}(M) \overset{\beta}{\longleftrightarrow} \mathbf{B}(M)$$

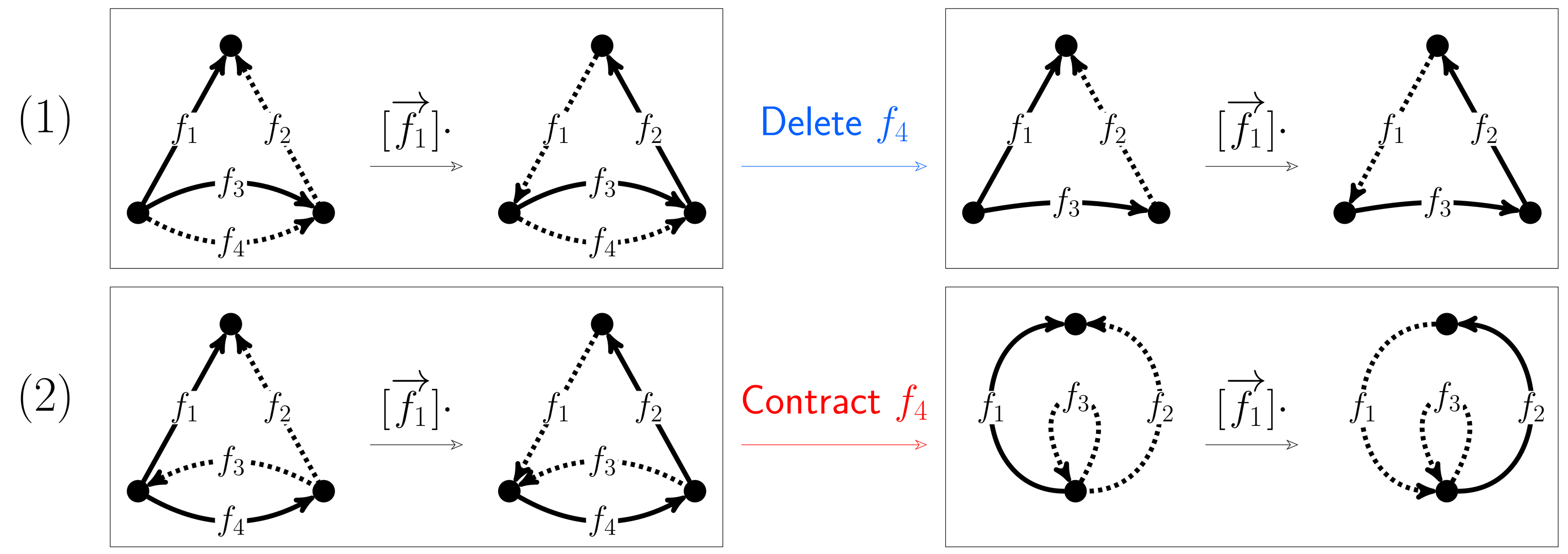
Theorem

The family of BBY actions is consistent.

More precisely, for the triple (M, σ, σ^*), suppose that $[\overrightarrow{f}]\cdot B_1 = B_2$, where $\overrightarrow{f}$ is a directed element and $B_1, B_2 \in \mathbf{B}(M)$. Then

1. $\forall e \in (B_1^c \cap B_2^c) \setminus f$, the BBY action for $(M\setminus e, \sigma\setminus e, \sigma^*\setminus e)$ satisfies $$[\overrightarrow{f}]\cdot(B_1\setminus e) = (B_2\setminus e);$$
2. $\forall e \in (B_1 \cap B_2) \setminus f$, the BBY action for $(M/e, \sigma/e, \sigma^*/e)$ satisfies $$[\overrightarrow{f}]\cdot(B_1/e) = (B_2/e).$$

Here the three $[\overrightarrow{f}]$'s are in $S(M)$, $S(M\setminus e)$, and $S(M/e)$, respectively,



Some Proof Ingredients

- We work with (σ,σ^*)-*compatible orientations* $\mathbf{O}$ as representatives of reversal classes, and show that the action of a directed element on $\mathcal{O} \in \mathbf{O}$ only modifies at most one circuit and one cocircuit, allowing a local analysis.
- We use the *fourientation* approach to describe BBY bijections, which provides some useful orthogonality results.

Corollary (Ganguly–McDonough 2023, Tóthmérész 2023)

The *rotor-routing* sandpile torsor algorithm on plane graphs is consistent.

Conjecture

BBY action induces the unique consistent sandpile torsor algorithm with respect to triangulating circuit-cocircuit signatures.

References

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