Crystals for variations of decomposition tableaux

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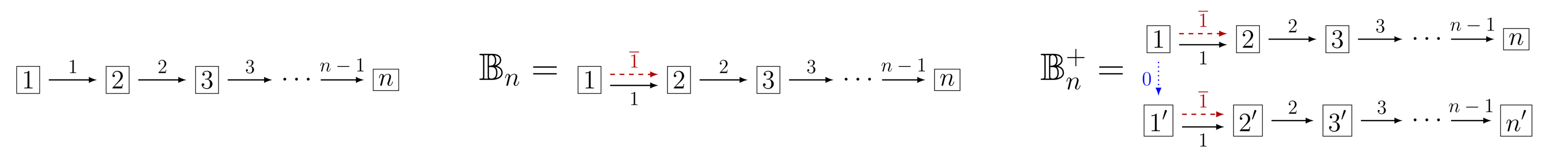
Abstract $\mathfrak{gl}_n$-, $\mathfrak{q}_n$- and $\mathfrak{q}_n^+$-crystals

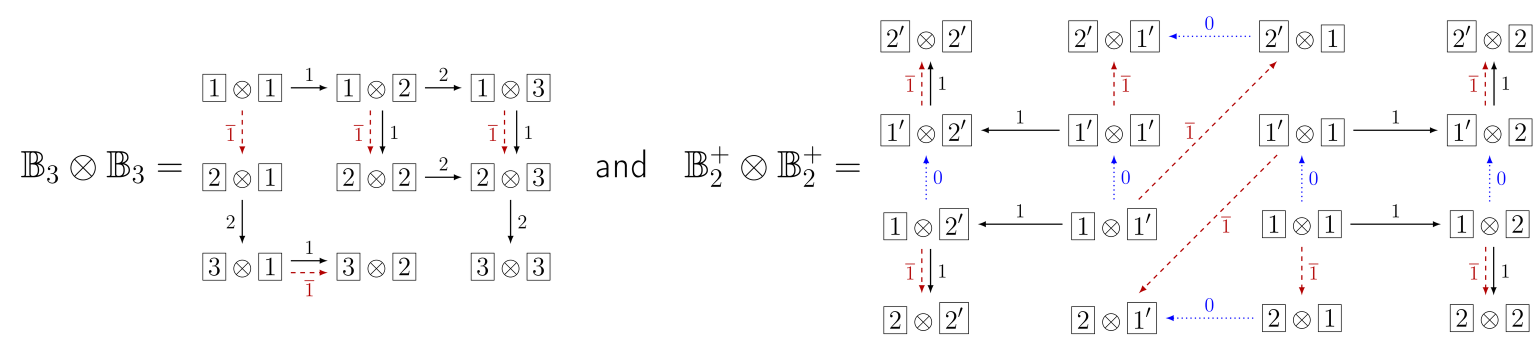
- A *$\mathfrak{gl}_n$-crystal* $\mathcal{B}$ is a DAG with edges labeled by $i \in [n-1]$ and a map $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$, satisfying certain axioms.
- A *$\mathfrak{q}_n$-crystal* is a $\mathfrak{gl}_n$-crystal with certain extra arrows labeled by $i = \overline{1}$, satisfying certain axioms.
- An *extended q_n-crystal* or *$\mathfrak{q}_n^+$-crystal* is $\mathfrak{q}_n$-crystal with extra arrows labeled by $i = 0$, satisfying certain conditions.
- For example, the *standard crystals* of type $\mathfrak{gl}_n$, $\mathfrak{q}_n$, and $\mathfrak{q}_n^+$ are respectively



The relevant weight maps all have $\mathrm{wt}(\boxed{i}) = \mathrm{wt}(\boxed{i'}) = \mathbf{e}_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{Z}^n$.

- There is a crystal structure on $\mathcal{B} \otimes \mathcal{C} = \{b \otimes c : b \in \mathcal{B}, c \in \mathcal{C}\}$ when $\mathcal{B}$ and $\mathcal{C}$ are both $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, or $\mathfrak{q}_n^+$-crystals.
- The $\mathfrak{q}_n$ and $\mathfrak{q}_n^+$ tensor products extend the $\mathfrak{gl}_n$ tensor product, but the $\mathfrak{q}_n^+$ case does **not** extend the $\mathfrak{q}_n$ one.
- The *character* of a finite crystal $\mathcal{B}$ is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x^{\mathrm{wt}(b)}$ and it always holds that $\mathrm{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$.

Tensor products and normal crystals



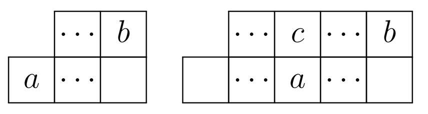
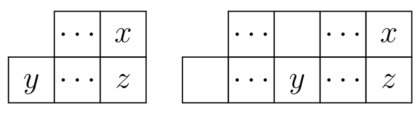
- A *normal* $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, or $\mathfrak{q}_n^+$-crystal is a component of a tensor power of the standard crystal, or a union of such crystals.

Theorem (Kashiwara; Grantcharov et al.; Marberg–T.)

Characters of normal $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, and $\mathfrak{q}_n^+$-crystals are respectively *Schur positive*, *Schur P-positive*, and *Schur Q-positive*.

Crystals on primed decomposition tableaux

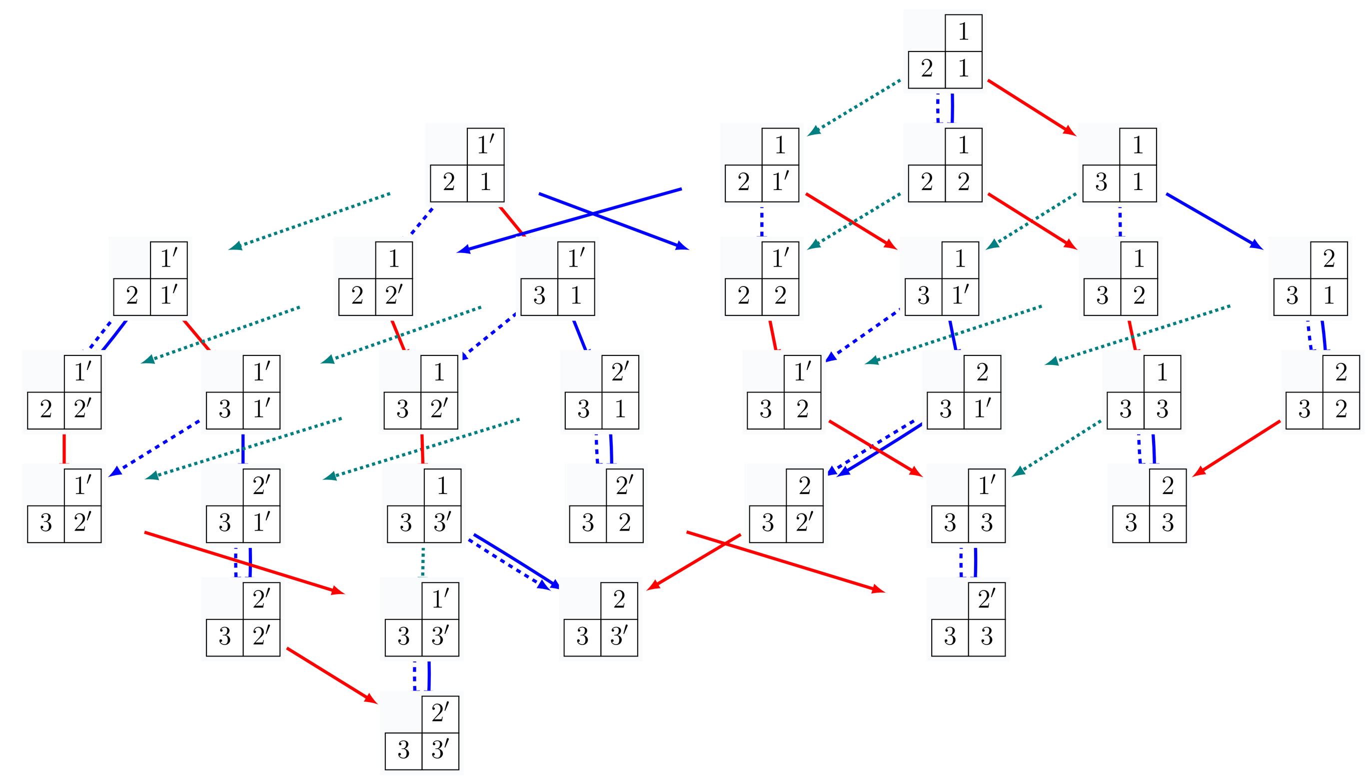
- Assume $\lambda = (\lambda_1 > \lambda_2 > \cdots > 0)$ is a strict integer partition. A *hook word* is a sequence of positive integers $w = w_1 w_2 \cdots w_N$ such that $w_1 \geq w_2 \geq \cdots \geq w_m < w_{m+1} < w_{m+2} < \cdots < w_N$ for some $1 \leq m \leq N$.
- A *(semistandard) decomposition tableau* of shape λ is a shifted tableau T of shape λ whose rows are hook words such that **none** of the following patterns occur in consecutive rows (drawing tableaux in French notation):

 for $a \leq b \leq c$ or  for $x < y < z$.

- Denote the set of decomposition tableau of shape λ with all entries $\leq n$ as $\mathsf{DecTab}_n(\lambda)$. Let $\mathsf{DecTab}_n^+(\lambda)$ be the set of tableaux formed by taking elements of $\mathsf{DecTab}_n(\lambda)$ and adding primes to the middle elements of zero or more rows.
- The *row reading word* of a shifted tableau T is the word $\mathbf{row}(T)$ formed by reading the rows from left to right, starting with the top row in French notation. The *reverse reading word* $\mathbf{revrow}(T)$ is the reversal of $\mathbf{row}(T)$.

Theorem (Marberg–T., 2023)

There is a unique (normal) $\mathfrak{q}_n^+$-crystal structure on $\mathsf{DecTab}_n^+(\lambda)$ that makes $\mathbf{revrow} : \mathsf{DecTab}_n^+(\lambda) \to (\mathbb{B}_n^+)^{\otimes|\lambda|}$ into a $\mathfrak{q}_n^+$-crystal embedding, where $\mathbb{B}_n^+$ denotes the standard $\mathfrak{q}_n^+$-crystal.



Theorem (Marberg–T., 2023)

Each connected normal $\mathfrak{q}_n^+$-crystal is isomorphic to $\mathsf{DecTab}_n^+(\lambda)$ for a unique λ. The crystal $\mathsf{DecTab}_n^+(\lambda)$ has a unique *$\mathfrak{q}_n^+$-highest weight element* of weight λ, and the character of an arbitrary normal $\mathfrak{q}_n^+$-crystal $\mathcal{B}$ has Schur Q-expansion

$$\mathrm{ch}(\mathcal{B}) = \textstyle\sum_b Q_{\mathrm{wt}(b)}(x_1, x_2, \dots, x_n) \quad \text{where the sum is over } \mathfrak{q}_n^+\text{-highest weight elements.}$$

Hence the coefficients in $Q_\lambda Q_\mu = \sum_\nu a_{\lambda\mu}^\nu Q_\nu$ count the $\mathfrak{q}_n^+$-highest weights in $\mathsf{DecTab}_n^+(\lambda) \otimes \mathsf{DecTab}_n^+(\mu)$ for $n \gg 0$.

Set-valued decomposition tableaux

- Let $\mathsf{SetShTab}^+(\lambda)$ be the set of *semistandard set-valued shifted tableaux* of shape λ. The entries of such tableaux are nonempty finite subsets of $\{1' < 1 < 2' < 2 < \dots\}$ that weakly increase along rows and columns in a specific sense.
- Let $\mathsf{SetShTab}(\lambda) \subseteq \mathsf{SetShTab}^+(\lambda)$ be the subset of tableaux with no primed numbers in any diagonal boxes.
- The *K-theoretic Schur P- and Q-functions* of λ are defined to be the power series

$$GP_\lambda = \sum_{T \in \mathsf{SetShTab}(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!] \quad \text{and} \quad GQ_\lambda = \sum_{T \in \mathsf{SetShTab}^+(\lambda)} x^T \in \mathbb{N}[\![x_1, x_2, \dots]\!].$$

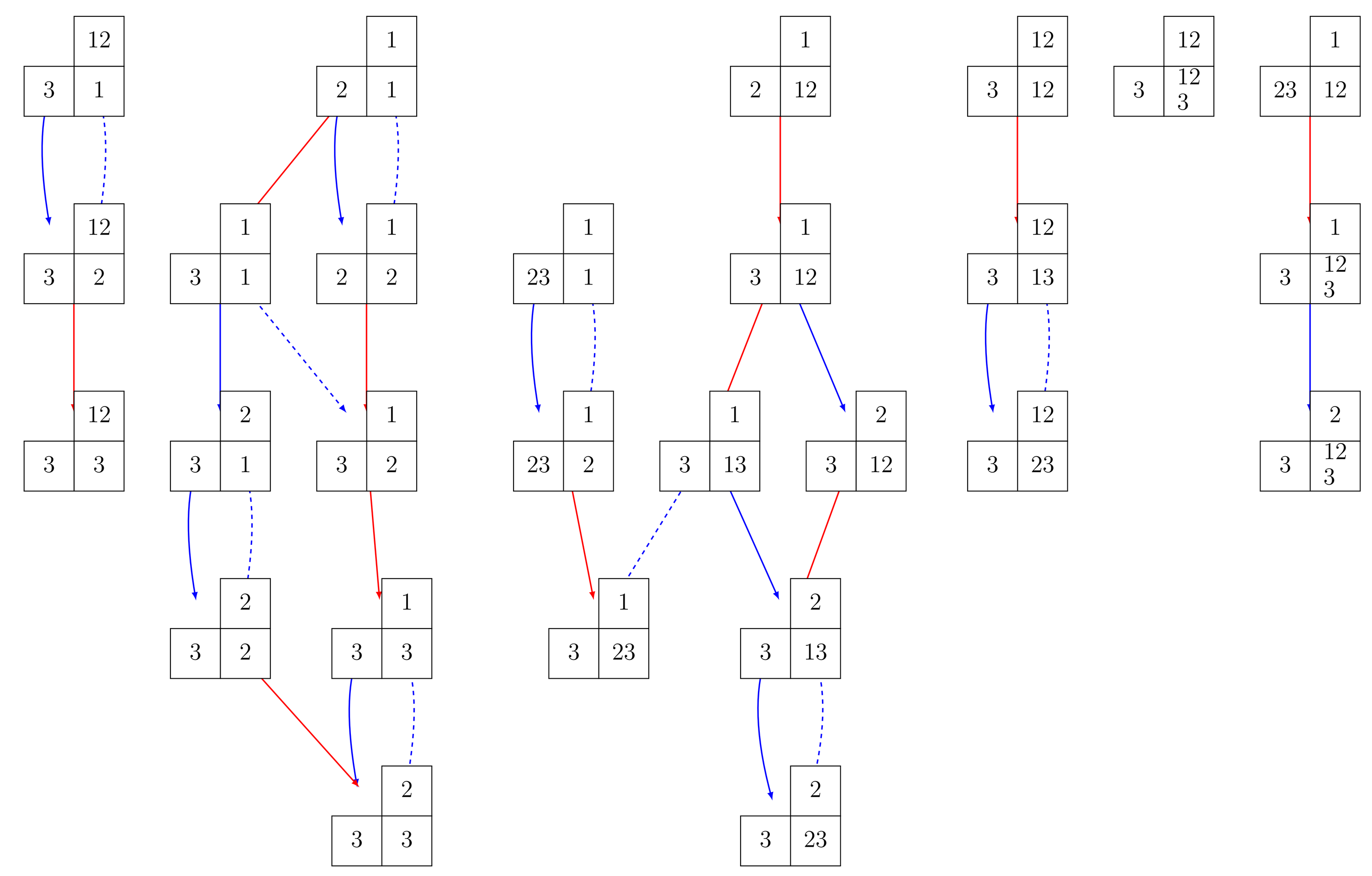
- Let $\mathsf{SetDecTab}(\lambda)$ be the set of all *(semistandard) set-valued decomposition tableaux* of shape λ.
A set-valued shifted tableaux of shape λ belongs to $\mathsf{SetDecTab}(\lambda)$ if all of its *distributions* are in $\mathsf{DecTab}(\lambda)$.

Conjecture (Cho–Ikeda)

If λ is any strict partition then $GP_\lambda = \sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$.

Theorem (Marberg–T., 2023)

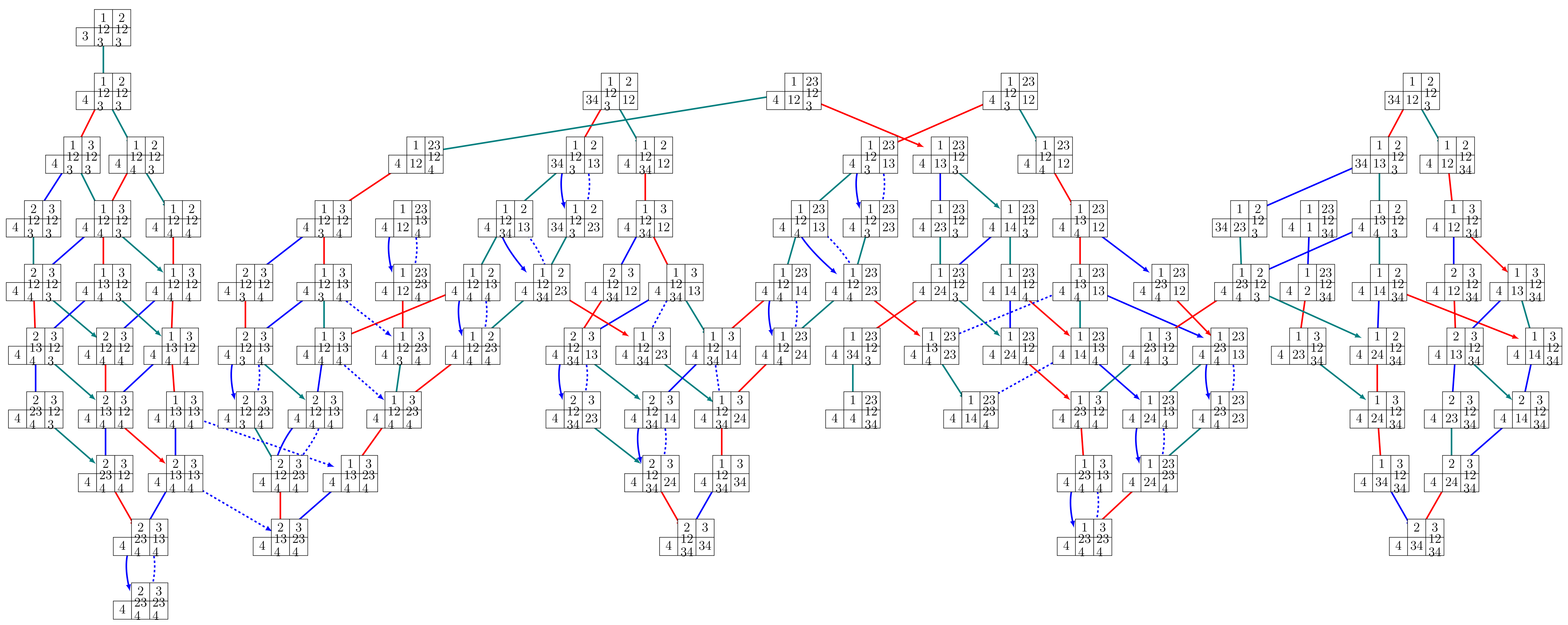
If λ is any strict partition then $\mathsf{SetDecTab}_n(\lambda)$ is naturally a $\mathfrak{q}_n$-crystal for certain explicit crystal operators.



Corollary (Marberg–T., 2023)

The power series $\sum_{T \in \mathsf{SetDecTab}(\lambda)} x^T$ is symmetric and equal to $GP_\lambda +$ (higher order GP-functions with $\mathbb{Z}$-coefficients).

Crystals on set-valued decomposition tableaux usually disconnected, generally not normal, can have multiple highest weights



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