# A signed $e$-expansion of the chromatic quasisymmetric function

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## Background

$G$: graph with vertex set $[n] = \{1, \dots, n\}$
$\kappa : [n] \to \mathbb{P}$ proper colouring: if $(i,j) \in E(G)$, then $\kappa(i) \neq \kappa(j)$
$\operatorname{asc}(\kappa)$: number of $(i,j) \in E(G)$ with $i < j$ and $\kappa(i) < \kappa(j)$

**Chromatic quasisymmetric function:**

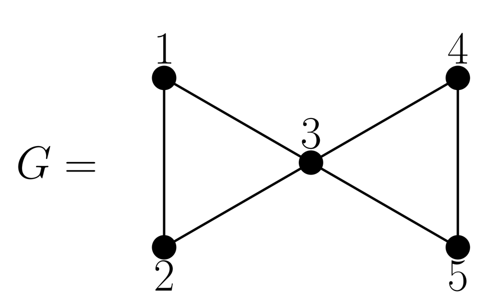
$$X_G(\boldsymbol{x};q) = \sum_{\kappa \text{ proper}} q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

natural unit interval graph: if $1 \leq i < j < k \leq n$ and $(i,k) \in E(G)$, then $(i,j),(j,k) \in E(G)$
elementary symmetric function:

$$e_\mu = e_{\mu_1} \cdots e_{\mu_\ell}, \text{ where } e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

**Theorem:** (Shareshian–Wachs 2016) If $G$ is a natural unit interval graph, then $X_G(\boldsymbol{x};q)$ is symmetric and palindromic in $q$.

**Example:**




$$X_G(\boldsymbol{x};q) = (q^2 + 2q^3 + q^4)e_{32} + (q + 3q^2 + 4q^3 + 3q^4 + q^5)e_{41} + (1 + 3q + 4q^2 + 4q^3 + 4q^4 + 3q^5 + q^6)e_5$$

**Conjecture:** (Stanley–Stembridge (1995), Shareshian–Wachs (2016)) If $G$ is a natural unit interval graph, then $X_G(\boldsymbol{x};q)$ is $e$-positive and $e$-unimodal.

**Example:** The complete graph $K_n$ has $X_{K_n}(\boldsymbol{x};q) = [n]_q! e_n$, where $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$ and $[k]_q = 1 + q + \cdots + q^{k-1}$.

## Forest triples

$T$ decreasing tree: no vertex has two larger neighbours.
$\sigma = \operatorname{list}(T)$: read vertices starting from $\min(T)$ and reading smallest vertex adjacent to read vertex at each step
A *tree triple* of $G$ is an object $\mathcal{T} = (T, \alpha, r)$, where

- $T$ is a decreasing subtree of $G$,
- $\alpha$ is an integer composition with $|\alpha| = |V(T)|$,
- $r$ is a positive integer with $r \leq \alpha_1$.

### Definition: Forest Triple of $G$

A *forest triple* of $G$ is a sequence of tree triples

$$\mathcal{F} = (\mathcal{T}_1 = (T_1, \alpha^{(1)}, r_1), \dots, \mathcal{T}_m = (T_m, \alpha^{(m)}, r_m))$$

such that $V(T_1) \sqcup \cdots \sqcup V(T_m) = [n]$ and

$$\min(V(T_1)) < \cdots < \min(V(T_m)).$$

- $\operatorname{type}(\mathcal{F}) = \operatorname{sort}(\alpha^{(1)} \cdots \alpha^{(m)})$
- $\operatorname{sign}(\mathcal{F}) = (-1)^{\sum_{i=1}^m (\ell(\alpha^{(i)})-1)}$
- $\operatorname{weight}(\mathcal{F}) = \operatorname{inv}_G(\operatorname{list}(T_1) \cdots \operatorname{list}(T_m)) + \sum_{i=1}^m (r_i - 1)$

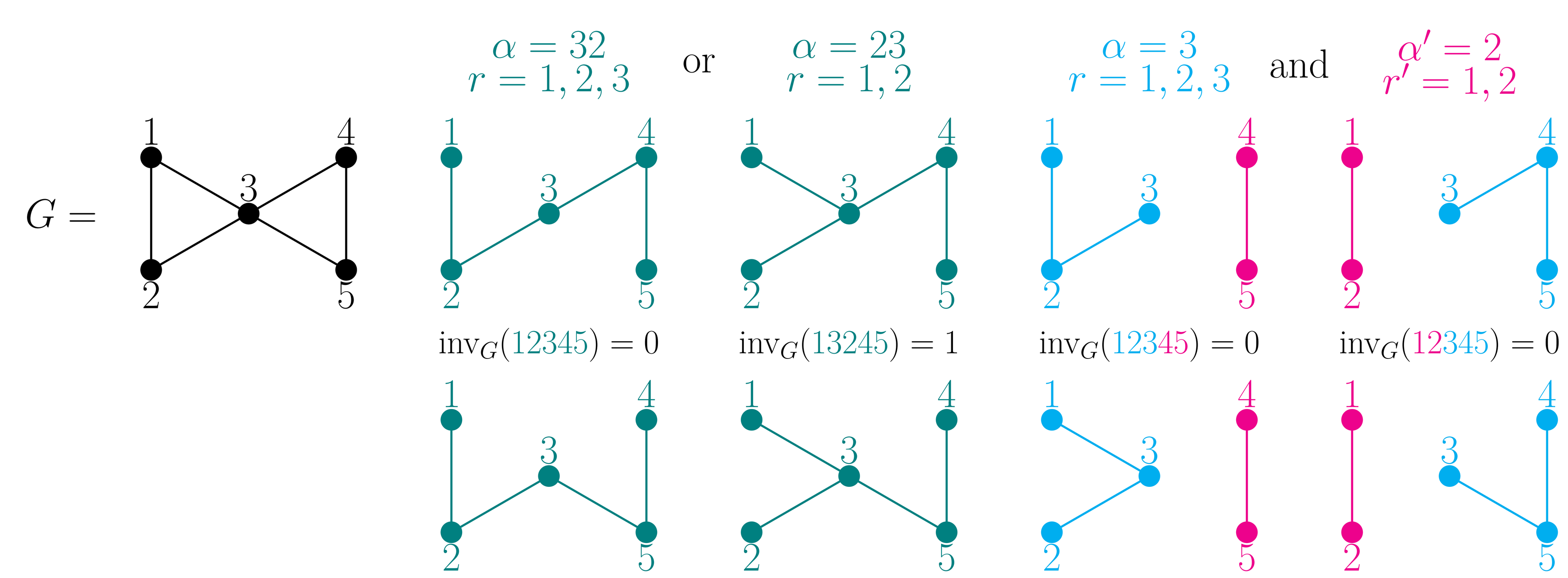
### Theorem: Signed $e$-expansion

If $G$ is a natural unit interval graph, then

$$X_G(\boldsymbol{x};q) = \sum_{\mathcal{F} \in \operatorname{FT}(G)} \operatorname{sign}(\mathcal{F}) q^{\operatorname{weight}(\mathcal{F})} e_{\operatorname{type}(\mathcal{F})} = \sum_\mu \left( \sum_{\mathcal{F} \in \operatorname{FT}_\mu(G)} \operatorname{sign}(\mathcal{F}) q^{\operatorname{weight}(\mathcal{F})} \right) e_\mu$$

## Example

The coefficient of $e_{32}$ in $X_G(\boldsymbol{x};q)$ is $q^2(1+q)^2$.



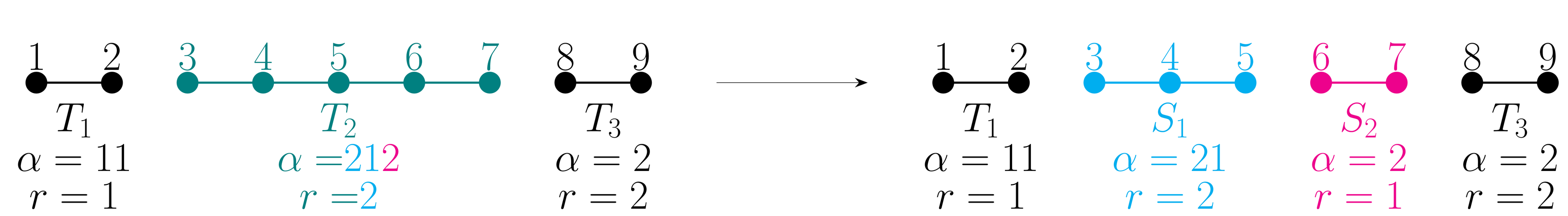

$$-(1 + 2q + q^2)(1 + q + q^2 + 1 + q) \qquad +(2 + 2q)(1 + q + q^2)(1 + q)$$

## Paths

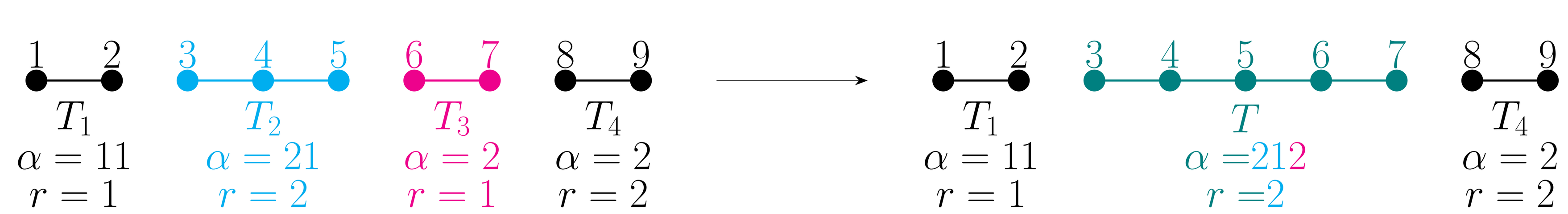
$$X_{P_n}(\boldsymbol{x};q) = \sum_{\alpha \vDash n} [\alpha_1]_q ([\alpha_2]_q - 1)([\alpha_3]_q - 1) \cdots ([\alpha_m]_q - 1) e_{\operatorname{sort}(\alpha)}.$$

**Proof:** We define a sign-reversing involution $\varphi$ on the set $\operatorname{FT}(P_n)$ of forest triples of $P_n$. Let $\mathcal{F} = (\mathcal{T}_1 = (P_{1 \to i_2 - 1}, \alpha^{(1)}, r_1), \dots, \mathcal{T}_m = (P_{i_m \to n}, \alpha^{(m)}, r_m))$ be a forest triple of $P_n$.
Let $j$ be maximal, if one exists, such that either:

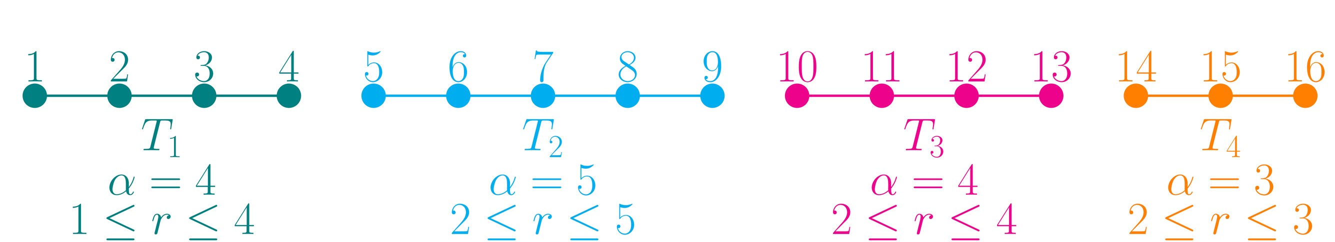
- $\ell(\alpha^{(j)}) \geq 2$: break $\mathcal{T}_j$ into $\mathcal{S}_1 = (P_{i_j \to i_{j+1} - 1 - \alpha_\ell^{(j)}}, \alpha^{(j)} \setminus \alpha_\ell^{(j)}, r_j)$, $\mathcal{S}_2 = (P_{i_{j+1} - \alpha_\ell^{(j)} \to i_{j+1} - 1}, \alpha_\ell^{(j)}, 1)$




- $j \geq 2$, $\ell(\alpha^{(j)}) = 1$, and $r_j = 1$: join $\mathcal{T}_{j-1}$ and $\mathcal{T}_j$ into $\mathcal{T} = (P_{i_{j-1} \to i_{j+1} - 1}, \alpha^{(j-1)} \cdot \alpha^{(j)}, r_{j-1})$



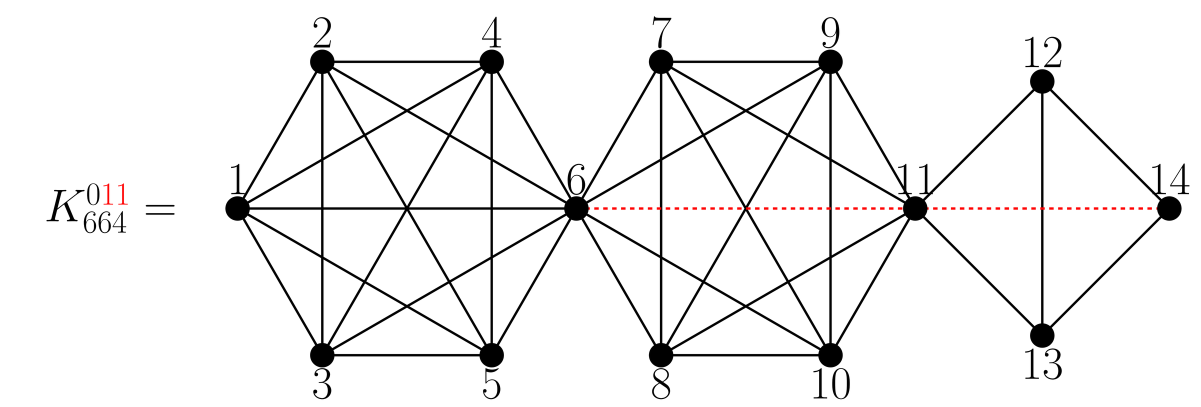

If no such $j$ exists, then $\mathcal{F}$ is a fixed point, meaning that $\varphi(\mathcal{F}) = \mathcal{F}$. Fixed points look like:




$\alpha = 4543$ $\qquad$ $4(5-1)(4-1)(3-1)$ choices of $r$

## Almost-$K$-chains

$\gamma = \gamma_1 \cdots \gamma_\ell$: sequence with every $\gamma_i \geq 2$
$\epsilon = \epsilon_1 \cdots \epsilon_\ell$: sequence with every $\epsilon_i \in \{0,1\}$
**Almost-$K$-chain $K_\gamma^\epsilon$:** graph formed by joining (almost)-cliques at single vertices




## Explicit formula

$A_\gamma^\epsilon$: set of weak compositions $\alpha = \alpha_1 \cdots \alpha_{\ell+1}$ of length $\ell + 1$ and size $n = |\gamma| - \ell + 1$ such that $\alpha_1 \geq 1$ and for each $2 \leq i \leq \ell + 1$ we have either

$$\alpha_i < \gamma_{i-1} - \epsilon_{i-1} \text{ and } \alpha_i + \cdots + \alpha_{\ell+1} < \gamma_i + \cdots + \gamma_\ell - (\ell - i), \text{ or}$$
$$\alpha_i \geq \gamma_{i-1} - \epsilon_{i-1} \text{ and } \alpha_i + \cdots + \alpha_{\ell+1} \geq \gamma_i + \cdots + \gamma_\ell - (\ell - i)$$

### Theorem: Formula for almost-$K$-chains

The chromatic quasisymmetric function of an almost-$K$-chain $K_\gamma^\epsilon$ is

$$X_{K_\gamma^\epsilon}(\boldsymbol{x};q) = [\gamma_1 - 2]_q! \cdots [\gamma_\ell - 2]_q! \sum_{\alpha \in A_\gamma^\epsilon} [\alpha_1]_q \prod_{i=2}^{\ell+1} q^{m_i} [|\alpha_i - (\gamma_{i-1} - 1 - \epsilon_{i-1})|]_q e_{\operatorname{sort}(\alpha)}.$$

In particular, $X_{K_\gamma^\epsilon}(\boldsymbol{x};q)$ is $e$-positive and $e$-unimodal.

**Corollary:** The chromatic quasisymmetric function of $K_{ab}$ is

$$X_{K_{ab}}(\boldsymbol{x};q) = [a-1]_q! [b-1]_q! \sum_{k=\max\{a,b\}}^{n} q^{n-k} [2k-n]_q e_{k(n-k)}$$

For example, we have

$$X_{K_{66}}(\boldsymbol{x};q) = [5]_q! [5]_q! (q^5 e_{65} + q^4 [3]_q e_{74} + q^3 [5]_q e_{83} + q^2 [7]_q e_{92} + q[9]_q e_{(10)1} + [11]_q e_{(11)})$$

## Further directions

**Problem:** Find sign-reversing involutions on forest triples for other natural unit interval graphs $G$ to prove that $X_G(\boldsymbol{x};q)$ is $e$-positive and $e$-unimodal.

**Problem:** Show that particular coefficients $c_\mu(q)$ are positive unimodal polynomials for every natural unit interval graph.

**Conjecture:** (Sagan–T. 2024+) If $G$ is a natural unit interval graph, then $X_G(\boldsymbol{x};q)$ is $e$-log-concave.

## References

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[6] F. Tom, *A signed e-expansion of the chromatic quasisymmetric function.* arXiv:2311.08020 (2024).