Stanley chromatic symmetric functions and a conjecture in the representation theory of unipotent groups

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$f(UT_\bullet; conj)$

$UT_n = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \;\middle|\; * \in \mathbb{F}_q \right\} \subseteq GL_n(\mathbb{F}_q)$

$\dim(f(UT_n; conj)) =$ 

set partitions of $\{1,2,\dots,n\}$ $\leq$  $\leq$ # simple labelled graphs on $\{1,2,\dots,n\}$

(q=2)

$U_{m,n}(\psi \otimes \chi) = \mathrm{Inf}_{UT_m \oplus UT_n}^{UT_{m+n}}(\psi \otimes \chi)$

where

$UT_m \oplus UT_n = \left[\begin{array}{c|c} UT_m & 0 \\ \hline 0 & UT_n \end{array}\right] \Leftarrow UT_{m+n}$

$D_{m,n}(\psi) = \sum_{\substack{A \subseteq \{1,2,\dots,m+n\} \\ |A| = m}} \mathrm{Res}^{UT_{m+n}}_{UT_A \times UT_{\bar{A}}}(\psi)$

where

$UT_A = \{u \in UT_{m+n} \mid (u - Id)_{ij} \neq 0 \Rightarrow i,j \in A\}$

Ex:

$UT_{\{1,2,4,6\}} = \begin{bmatrix} 1 & * & 0 & * & 0 & * \\ & 1 & 0 & * & 0 & * \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & * \\ & 0 & & & 1 & 0 \\ & & & & & 1 \end{bmatrix} \subseteq UT_6$

Two favorite bases:

- Equivalence class identifier functions
- Orthogonal character basis

$\subseteq f(UT_\bullet)$

$\dim(f(UT_n))$

$\updownarrow$

graphs with vertices $\{1,2,\dots,n\}$

graded

connected

noncommutative

cocommutative

$\supseteq f(UT_\bullet; super)$

super

$u \sim v$ iff $u - Id = a(v - Id)b + Id$, $a, b \in UT_n$

$\dim(f(S_n; super)) =$ # set partitions of $\{1,2,\dots,n\}$

$f(UT_n; super)$ has a basis of orthogonal characters $\{\chi^\lambda \mid \lambda \text{ a set partition}\}$

a sub Hopf algebra of $f(UT_\bullet; conj)$

[Diaconis–Isaacs]. χ^λ is an irreducible character if and only if λ is noncrossing.

Combinatorial Hopf algebras

A combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ is a pair, where $\mathcal{H}$ is a connected graded Hopf algebra, and $\zeta: \mathcal{H} \to \mathbb{C}$ is an algebra morphism.

Thm[Aguiar–Bergeron–Sotille] If $(\mathcal{H}, \zeta)$ is a cocommutative combinatorial Hopf algebra, then there exists a canonical Hopf algebra isomorphism

$ch_\zeta : \mathcal{H} \longrightarrow \mathrm{Sym}.$

Favorite setup: Let $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ be a tower of finite groups. For each G_n define an equivalence relation (such as conjugacy), and let $\sim$ denote the combined relation on $\bigcup_{n \geq 0} G_n$.

Define $f(G_\bullet; \sim) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} f(G_n; \sim)$,

where

$f(G_n; \sim) = \{\psi : G_n \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}.$

Given (appropriate) functions

$U_{m,n} : f(G_m; \sim) \otimes f(G_n; \sim) \longrightarrow f(G_{m+n}; \sim)$

$D_{m,n} : f(G_{m+n}; \sim) \longrightarrow f(G_m; \sim) \otimes f(G_n; \sim)$

define a product and coproduct by

$\chi \cdot \psi = U_{m,n}(\chi \otimes \psi)$

$\Delta(\psi) = \sum_{j=0}^{n} D_{j,n-j}(\psi)$

$\chi \in f(G_m; \sim)$, $\psi \in f(G_n; \sim)$

to get a Hopf algebra.

Our algebra morphism will be given by

$\langle \mathbb{1}_\bullet \rangle : f(G_n; \sim) \longrightarrow \mathbb{C}$

$\psi \longmapsto \langle \psi, \mathbb{1}_n \rangle_{G_n}$

average value of the function

trivial character

By definition, this will be integral on characters.

$f(S_\bullet; conj)$

$S_n = \left\{ \begin{array}{l} \text{permutations} \\ \text{of } \{1,2,\dots,n\} \end{array} \right\}$

$U_{m,n}(\chi \otimes \psi) = \mathrm{Ind}^{S_{m+n}}_{S_m \times S_n}(\chi \otimes \psi)$

$D_{m,n}(\chi) = \mathrm{Res}^{S_{m+n}}_{S_{m+n}}(\chi)$

$ch_{\langle \mathbb{1}_\bullet \rangle}(\psi^\lambda) = s_\lambda$

$ch^*_{\langle \mathbb{1}_\bullet \rangle}(\psi^\lambda) = \sum_{w \in S_{\ell(\lambda)}} (-1)^{\ell(w)} \sum_{\substack{\underline{A} \models \{1,2,\dots,n\} \\ |A_j| = \lambda_j - j + w(j)}} \mathrm{Ind}^{UT_n}_{UT_{\underline{A}}}(\mathbb{1})$

irreducible character of S_n

We want $ch^*_{\langle \mathbb{1}_\bullet \rangle}$ and

NOT

$\mathrm{Res}^{f(UT_\bullet; conj)}_{f(UT_\bullet; super)}(ch_{\langle \mathbb{1}_\bullet \rangle})^*$

$f(UT_\bullet; conj)^*$

same underlying space

$U^*_{m,n}(\chi \otimes \psi) = \sum_{\substack{A \subseteq \{1,2,\dots,m+n\} \\ |A| = m}} \mathrm{Ind}^{UT_{m+n}}_{UT_A \times UT_{\bar{A}}}(\chi \otimes \psi)$

$ch^*_{\langle \mathbb{1}_\bullet \rangle}(e_n) = (-1)^{n-1} S(\mathbb{1})$

antipode

$ch^*_{\langle \mathbb{1}_\bullet \rangle}(h_n) = \mathbb{1}_n$

$D^*_{m,n}(\psi) = \mathrm{Def}^{UT_{m+n}}_{UT_m \oplus UT_n}(\psi)$

$-\det(h_{\lambda_i - i + j})$

$(\lambda_w)_i = \lambda_i - i + w(i)$

$ch^*_{\langle \mathbb{1}_\bullet \rangle}(s_\lambda) = \sum_{\substack{\underline{A} \models \{1,2,\dots,n\} \\ w \in S_n \\ \lambda_w = bl(\underline{A})}} (-1)^{\ell(w)-1} \mathrm{Ind}^{UT_n}_{UT_{\underline{A}}}(\mathbb{1})$

not clear how to compute

Stanley chromatic symmetric functions

$X_{(V,E)} = \sum_{\substack{c: V \to \mathbb{Z}_{\geq 1} \\ \text{properly colors } (V,E)}} x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} \cdots$

graph

where c properly colors (V,E) if $c(u) = c(v)$ implies $\{u,v\} \notin E$.

Q. When is $X_{(V,E)}$ s- or e-positive?

Thm[A-T]. If λ is non-nesting and noncrossing, then $ch_{\langle \mathbb{1}_\bullet \rangle}(\chi^\lambda) = \sum_{E \in \mathcal{V}} (q-1)^{s(E)} X_{(\{1,2,\dots,n\},E)}$.

a set of unit interval graphs

Conj.

There exist adjoint functors

$F : UT_n\text{-mod} \longrightarrow S_n\text{-mod}$

$F^* : S_n\text{-mod} \longrightarrow UT_n\text{-mod}$

such that

$$\begin{array}{ccccc} UT_n\text{-mod} & \xrightarrow{F} & S_n\text{-mod} & \xrightarrow{F^*} & UT_n\text{-mod} \\ \downarrow \text{trace} & & \downarrow \text{trace} & & \downarrow \text{trace} \\ f(UT_n, conj) & \xrightarrow{ch_{\langle \mathbb{1}_\bullet \rangle}} & f(S_n; conj) & \xrightarrow{ch^*_{\langle \mathbb{1}_\bullet \rangle}} & f(UT_n; conj) \end{array}$$

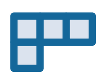
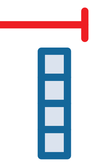
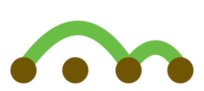
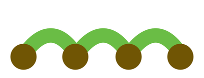
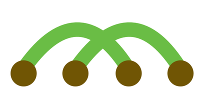
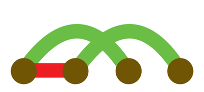
commute.

Some Evidence

$t = q - 1$

Coeff of s_μ in $ch_{\langle \mathbb{1}_\bullet \rangle}(\chi^\lambda)$

Coeff of e_μ in $ch_{\langle \mathbb{1}_\bullet \rangle}(\chi^\lambda)$

irreducible of UT_4					
	0 $8q^2t^2(3t+4)$	$2q^2$ $2q^2t(5t+6)$	$2q^2(2t+1)$ $4q^2t$	$2q^2(5t^2+8t+2)$ $2q^2$	$2q^2(12t^3+21t^2+8t+1)$ 0
	0 $8qt$	q $q(4t+3)$	q 0	$q(4t+5)$ 1	$q(12t+4)$ 0
	0 4	0 2	2 2	4 0	8 0
	0 $16t$	0 $2t$	4 4	$2(t+2)$ 0	$2(9t+2)$ 0
	0 $4(4t-1)$	0 $2(t+2)$	2 2	$2(t+3)$ 0	$2(9t+1)$ 0

Some Evidence*

Permutation characters: $ch^*(h_\lambda) = \sum_{\substack{\underline{A} \models \{1,2,\dots,n\} \\ bl(\underline{A}) = \lambda}} \mathrm{Ind}^{UT_n}_{UT_{\underline{A}}}(\mathbb{1})$ (by construction)

Coefficient of the trivial character: $ch^*(s_\lambda) = \psi^\lambda(1)\mathbb{1} + \cdots$ the irreducible character corresponding to λ

The sign character: $ch^*(e_n) = \sum_{\underline{A} \models \{1,2,\dots,n\}} (-1)^{n-\ell(\underline{A})} \mathrm{Ind}^{UT_n}_{UT_{\underline{A}}}(\mathbb{1})$

a priori, a virtual character

$= \mathbb{1} + \sum_{\substack{1 \leq i < j \leq n \\ \alpha \in \mathbb{F}_q^\times}} \left(\sum_{m=0}^{j-i-1} (m+2)! \binom{j-i-1}{m} t^m \right) \chi^{i \overset{\alpha}{\frown} j} + \cdots$