WEAK BRUHAT INTERVAL 0-HECKE MODULES IN FINITE TYPE

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COXETER GROUPS

Let (W, S) be a finite Coxeter system.

- $s \in S$ is a **left descent** of w if $\ell(sw) < \ell(w)$, $s \in S$ is a **right descent** of w if $\ell(ws) < \ell(w)$.
- Let $\mathcal{D}_L(w)$ denote the set of left descents of w, and let $\mathcal{D}_R(w)$ denote the set of right descents of w.
- For $I \subseteq S$, the **right descent class** $\mathcal{D}_I$ is the set $\{w \in W : \mathcal{D}_R(w) = I\}$.
- For $I \subseteq J \subseteq S$, let $\mathcal{D}_I^J$ denote the union of all $\mathcal{D}_X$ such that $I \subseteq X \subseteq J$.
- Let u_I denote the shortest element in $\mathcal{D}_I$ and v_I the longest element in $\mathcal{D}_I$.
- The **left weak Bruhat order** $\leq_L$ on W is defined by $u \leq_L v$ if some reduced word for u is a terminal segment in some reduced word for v.
- If $u \leq_L v$, the **left weak Bruhat interval** $[u, v]_L$ is the set $\{w \in W : u \leq_L w \leq_L v\}$.

0-HECKE ALGEBRAS

Let $\mathbb{K}$ be a field. The 0-**Hecke algebra** $H_W(0)$ is the $\mathbb{K}$-algebra generated by $\{\pi_s : s \in S\}$ with relations

$$\pi_s^2 = \pi_s \quad \text{and} \quad (\pi_s\pi_t)_{m_{st}} = (\pi_t\pi_s)_{m_{st}}$$

where $(ab)_r$ denotes the word $\cdots ab$ of length r alternating between a and b, and m_{st} is the order of st in W.

LINEAR OPERATORS ON $\mathbb{K}[u, v]_L$

Define linear operators $\{\pi_s : s \in S\}$ on the vector space $\mathbb{K}[u, v]_L$ by

$$\pi_s w = \begin{cases} w & \text{if } s \in \mathrm{D}_L(w), \\ sw & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \in [u, v]_L, \\ 0 & \text{if } s \notin \mathrm{D}_L(w) \text{ and } sw \notin [u, v]_L. \end{cases}$$

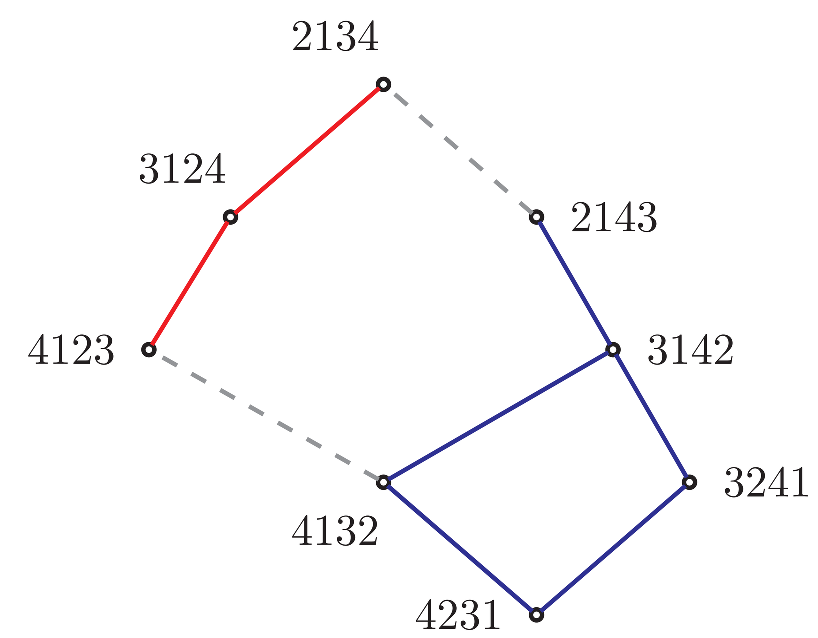
WEAK BRUHAT INTERVAL MODULES

Theorem: The operators $\{\pi_s : s \in S\}$ define an action of $H_W(0)$ on $\mathbb{K}[u, v]_L$.

Following Jung, Kim, Lee and Oh, who introduced and studied these modules in type A, we call the $H_W(0)$-module $\mathbb{K}[u, v]_L$ a **weak Bruhat interval module**, and denote it by $\mathrm{B}(u, v)$.

Theorem: Let P_I denote $\mathrm{B}(u_I, v_I)$ and P_I^J denote $\mathrm{B}(u_I, v_J)$. The P_I are projective indecomposable modules, and

$$\mathrm{P}_I^J \cong \bigoplus_{I \subseteq X \subseteq J} \mathrm{P}_X.$$



In S_4, $\mathrm{B}(2134, 4231) = \mathrm{P}_{\{1\}}^{\{1,3\}} \cong \mathrm{P}_{\{1\}} \oplus \mathrm{P}_{\{1,3\}}$.

STRUCTURAL RESULTS

Proposition: The weak Bruhat interval modules $\mathrm{B}(w, v_I)$ and $\mathrm{B}(u_I, w)$ are indecomposable for all $w \in \mathcal{D}_I$, and all submodules of $\mathrm{B}(w, v_I)$ and quotients of $\mathrm{B}(u_I, w)$ are also indecomposable.

Fayers introduced certain (dual) equivalences of the category $H_W(0)$-mod, based on involutions ϕ, θ and an anti-involution χ on $H_W(0)$ defined by

$$\phi : \pi_s \mapsto \pi_{w_0 s w_0}, \qquad \theta : \pi_s \mapsto 1 - \pi_s, \qquad \chi : \pi_s \mapsto \pi_s.$$

Let M be a $H_W(0)$-module. Define $H_W(0)$-modules $\phi[M]$, $\theta[M]$, $\chi[M]$ such that

- $\phi[M]$ has underlying space M, with action $\pi_s \cdot_\phi m = \phi(\pi_s) \cdot m$ for $m \in M$
- $\theta[M]$ has underlying space M, with action $\pi_s \cdot_\theta m = \theta(\pi_s) \cdot m$ for $m \in M$
- $\chi[M]$ has underlying space M^*, with action $(\pi_s \cdot^\chi f)(m) = f(\chi(\pi_s) \cdot m)$ for $f \in M^*$ and $m \in M$.

Theorem: Let $\hat{\theta} := \theta \circ \chi$ and $\hat{\omega} := \phi \circ \theta \circ \chi$. If Y is an upper order ideal in $[u, v]_L$, then we have the following isomorphisms of $H_W(0)$-modules.

$$\begin{aligned} \phi[\mathrm{B}(u, v)/\mathbb{K}Y] &\cong \mathbb{K}([w_0uw_0, w_0vw_0]_L \setminus w_0Yw_0), \\ \hat{\theta}[\mathrm{B}(u, v)/\mathbb{K}Y] &\cong \mathbb{K}([vw_0, uw_0]_L \setminus Yw_0), \\ \hat{\omega}[\mathrm{B}(u, v)/\mathbb{K}Y] &\cong \mathbb{K}([w_0v, w_0u]_L \setminus w_0Y). \end{aligned}$$

This extends certain results of Jung, Kim, Lee and Oh in type A, and is useful in establishing the injective hull results below.

Theorem: Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $u_J \notin Y$. Then P_I^J is the projective cover of $\mathrm{P}_I^J/\mathbb{K}Y$.

Theorem: Let Y be an upper order ideal in $\mathcal{D}_I^J$ with $v_I \in Y$. Then P_I^J is the injective hull of $\mathbb{K}Y$.

Specialising these results to Bruhat interval modules, we obtain

Corollary: Let $I \subseteq J$ and $w \in \mathcal{D}_J$. Then P_I^J is the projective cover of $\mathrm{B}(u_I, w)$.

Corollary: Let $I \subseteq J$ and $w \in \mathcal{D}_I$. Then P_I^J is the injective hull of $\mathrm{B}(w, v_J)$.

APPLICATIONS

The *quasisymmetric characteristic map* connects representation theory of type A 0-Hecke algebras to the algebra of quasisymmetric functions.

For several notable bases of quasisymmetric functions, modules of type A 0-Hecke algebras have been constructed such that their quasisymmetric characteristics are elements of these bases. These include

- Dual immaculate functions
- Extended Schur functions
- Row-strict dual immaculate functions
- Row-strict extended Schur functions
- Quasisymmetric Schur functions
- Young row-strict quasisymmetric Schur functions.

The structural results on weak Bruhat interval modules can be used to recover a number of results on indecomposability and projective covers, and obtain new results, for some of these modules in a uniform manner.

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