Chain polynomials of generalized paving matroids

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Background and aim

We study some posets whose chain polynomials have only real roots. These polynomials have been studied and proven to be real rooted for some interesting classes of posets such as the face lattices of simplicial polytopes [3] and cubical polytopes [1], subspace lattices and partition lattices of types A and B [2]. However, there are examples of finite distributive lattices with chain polynomials whose roots are not all real [4]. In [2], Athanasiadis and Kalampogia-Evangelinou conjectured that the chain polynomials of geometric lattices are real-rooted. We investigate this claim and provide some partial results.

Chain polynomials

The chain polynomial of a finite poset P is defined as

$$c_P(t) := \sum_{k\geq 0} c_k(P)t^k,$$

where $c_k(P)$ is the number of k-element chains in P.

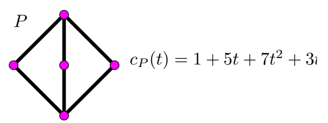

Figure 1: Chain polynomial of the poset P.

Definition (Geometric lattice). A *geometric lattice* is the lattice ordered by inclusion whose elements are given by $\bigcap_{H\in\mathcal{C}} H$, where $\mathcal{C}$ is a collection of hyperplanes of some matroid $(E, \mathcal{H})$. Each element of a geometric lattice is called a *flat*.

Conjecture (Athanasiadis-Kalampogia-Evangelinou, 2023). The chain polynomial $c_{\mathcal{L}}(t)$ is real-rooted for every geometric lattice $\mathcal{L}$.

Paving matroids

Definition (d-partition). A family $\mathcal{F}$ of two or more sets forms a d-partition if every set in $\mathcal{F}$ has size at least d, and every d-element subset of $\bigcup \mathcal{F}$ is a subset of exactly one set in $\mathcal{F}$.

Definition (Paving matroid). A matroid $M = (E, \mathcal{H})$ of rank $d+1$ is called a paving matroid if $\mathcal{H}$ forms a d-partition of E.

Conjecture (Mayhew-Newman-Welsh-Whittle, 2011). Almost all matroids are paving matroids, that is,

$$\lim_{n\to\infty}\left(\frac{\text{number of paving matroids on } [n]}{\text{number of matroids on } [n]}\right) = 1.$$

Theorem (Brändén-Saud, 2024). The chain polynomial of a paving matroid is real-rooted.

Generalized paving matroids

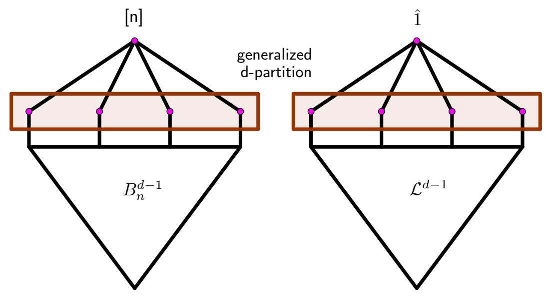

Figure 2: Construction of generalized paving matroids.

Theorem. Let M be a matroid such that its lattice of flats $\mathcal{L}$ has the following property: if $\rho(x) = \rho(u)$ and $\rho(y) = \rho(v)$, then $[x, y]$ and $[u, v]$ are isomorphic. Then the chain polynomial of the lattice of flats of M is real-rooted. In particular, the chain polynomial of any $\mathcal{L}$-paving matroid is real-rooted.

Example. $\mathcal{L}_q^d$ is a geometric lattice with the property above. In particular, $\mathcal{L}_q^d$-paving matroids have real-rooted chain polynomials.

References

[1] Christos A Athanasiadis. Face numbers of barycentric subdivisions of cubical complexes. Israel Journal of Mathematics, 246:423–439, 2021.

[2] Christos A Athanasiadis and Katerina Kalampogia-Evangelinou. Chain enumeration, partition lattices and polynomials with only real roots. Combinatorial Theory, 3 (1), 2023.

[3] Francesco Brenti and Volkmar Welker. f-vectors of barycentric subdivisions. Mathematische Zeitschrift, 259:849–865, 2008.

[4] John Stembridge. Counterexamples to the poset conjectures of neggers, stanley, and stembridge. Transactions of the American Mathematical Society, 359(3):1115–1128, 2007.

Single-element extensions

A matroid N with ground set E is called an extension of a matroid M if $M = N\backslash T$, where $T \subset E$ and

$$\{B : B \text{ is a basis of } N \text{ and } J \not\subset E\}.$$

When $|T| = 1$, N is called a single-element extension of $N\backslash T$. A modular cut $\mathcal{M}$ of a matroid M is a set of flats of M that satisfies the following:

- if $F \in \mathcal{M}$ and F' is a flat of M containing F, then $F' \in \mathcal{M}$;
- if $F_1, F_2 \in \mathcal{M}$ and $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{M}$.

There is a one-to-one correspondence between the modular cuts of a matroid M and single-element extensions of M, denoted by $M +_{\mathcal{M}} 4$ hence all matroids come from single-element extensions of a uniform matroid U_n^n.

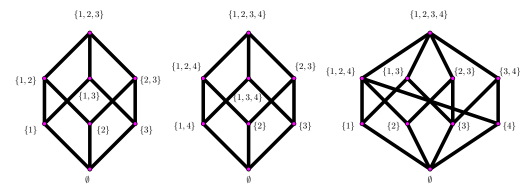

Figure 3: The Hasse diagramas of U_3^3, $U_3^3 +_{\mathcal{M}_1} 4$ and $U_3^3 +_{\mathcal{M}_2} 4$, respectively, where $\mathcal{M}_1 = [[1], [3]]$ and $\mathcal{M}_2 = [[2], [3]]$.

Theorem. If the h-polynomials of the order complexes of the posets X and Y have nonnegative coefficients, then the chain polynomial of $X \times Y$ is real-rooted.

Corollary. The chain polynomial of $U_n^n +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^n.

Corollary. The chain polynomial of $U_n^{n-1} +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^{n-1}.