Higher Specht Polynomials and Tableaux Bijections for Hessenberg Varieties

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Combinatorial Background

- A **Hessenberg function** is $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ that is weakly increasing and $h(i) \geq i$ for all i. We use the shorthand $h = (h(1), h(2), \ldots, h(n))$. Hessenberg functions can be visualized via Dyck paths, as seen below.
- The poset P_h on $\{1, 2, \ldots, n\}$ is defined by $i <_{P_h} j$ whenever $h(i) < j$.
- The graph G_h is the incomparability graph of P_h.

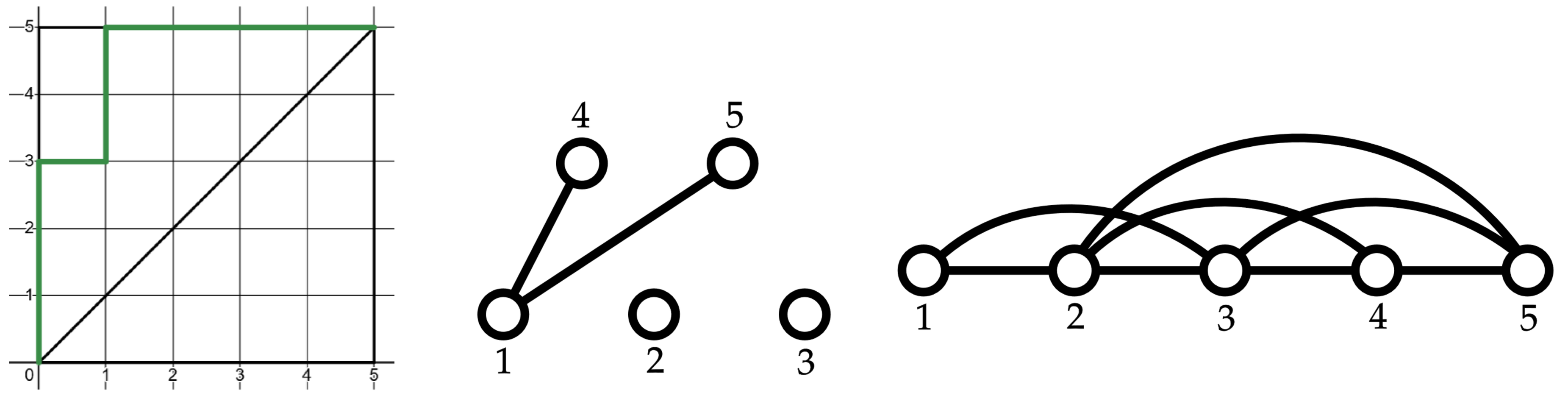


Figure: For $h = (3, 5, 5, 5, 5)$, the corresponding Dyck path, poset P_h, and graph G_h.

- The **chromatic quasisymmetric function** [5] of a graph G is $X_G(x; q) = \sum_{\kappa} q^{\operatorname{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$ where the sum is over all proper colorings of the vertices of G, and $\operatorname{asc}(\kappa)$ is the number of pairs of vertices $i < j$ such that $\kappa(i) < \kappa(j)$.

Geometric Background

- The **flag variety** $\mathrm{Flag}(\mathbb{C}^n)$ is the set of nested subspaces $F_\bullet = F_0 \subset F_1 \subset \cdots \subset F_n$, called **flags**, where each F_i is a subspace of $\mathbb{C}^n$ of dimension i.
- Given a linear map $S : \mathbb{C}^n \to \mathbb{C}^n$, and a Hessenberg function h, the corresponding **Hessenberg variety** is

$$\mathrm{Hess}(S, h) = \{F_\bullet \in \mathrm{Flag}(\mathbb{C}^n) \mid S(F_i) \subseteq F_{h(i)}\}$$

- The cohomology ring $H^*(\mathrm{Hess}(S, h))$ is a graded $\mathfrak{S}_n$-module [7], and the image under the Frobenius character map is related to the chromatic quasisymmetric function in the following way. [2, 4] If S is a regular semisimple map, then we have

$$\sum_{k=0}^{N} \mathrm{Frob}(H^{2k}(\mathrm{Hess}(S, h)))\, q^k = \omega\, X_{G_h}(x; q)$$

- In [1], the authors describe $H^*(\mathrm{Hess}(S, h))$ as a polynomial ring when $h = (h(1), n, \ldots, n)$:

$$H^*(\mathrm{Hess}(S, h)) \cong \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I$$

where the generators of I are explicitly given.
- In most cases, however, there is not a polynomial ring realization of $H^*(\mathrm{Hess}(S, h))$.

Higher Specht basis for $H^*(\mathrm{Hess}(S, h))$

- Suppose that $h = (h(1), n, \ldots, n)$.
- We begin by adapting the basis in [1] to form a higher Specht basis. Define sets:

$$B_1 = \left\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \text{ not containing the factor } \prod_{\ell=1}^{h(1)} x_\ell \right\}$$

$$B_2 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_k - y_1) \text{ not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\}$$

where $0 \leq i_j \leq n - j$ in B_1 and $0 \leq \ell_j \leq n - j - 1$ and $2 \leq k \leq n$ in B_2.
- $\mathfrak{S}_n$ acts on these sets by fixing the x_i's and permuting the y_i's.

Theorem 1. *The set $B_1 \cup B_2$ forms a higher Specht basis of $H^*(\mathrm{Hess}(S, h))$. In particular, if this $\mathfrak{S}_n$-module decomposes into irreducibles as*

$$H^*(\mathrm{Hess}(S, h)) \cong \bigoplus_{\lambda \vdash n} c_\lambda V_\lambda, \quad \text{then we have} \quad B_1 \cup B_2 = \bigcup_{\lambda \vdash n} \bigcup_{i=1}^{c_\lambda} B_{i,\lambda}$$

where the elements of $B_{i,\lambda}$ form a basis of the i-th copy of V_λ in the decomposition.

Corollary 2. *The $\mathfrak{S}_n$-module $H^*(\mathrm{Hess}(S, h))$ decomposes into $h(1)(n-1)!$ copies of the trivial representation and $(n - h(1))(n-2)!$ copies of the standard representation.*

Theorem 3. *The following set of monomials form a (permutation) basis of $H^*(\mathrm{Hess}(S, h))$.*

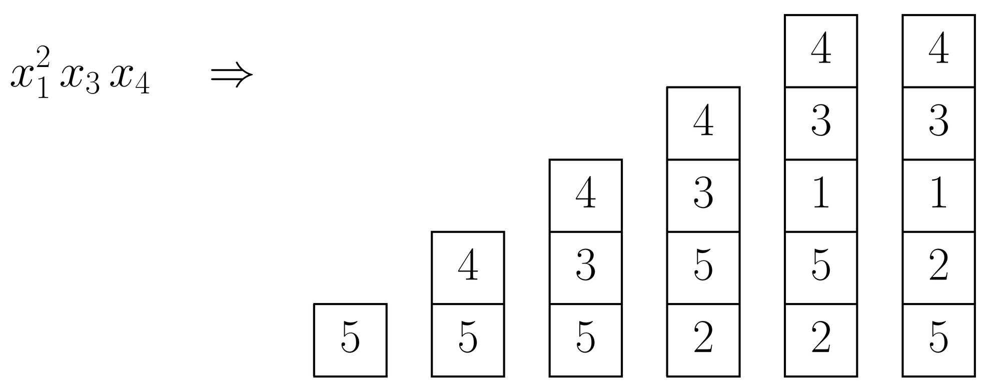
$$B_3 = \left\{ x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \text{ not containing the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \right\}$$

where $1 \leq k \leq n$.

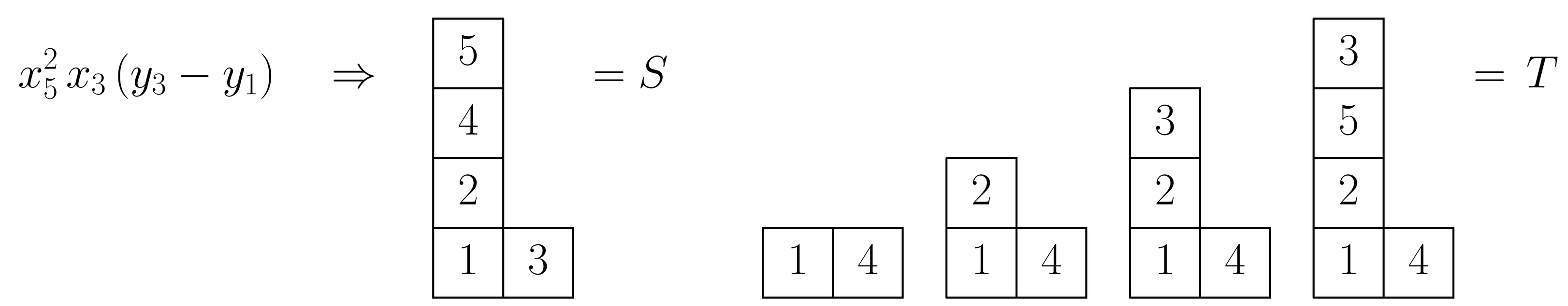
Bijections with P_h-tableaux

- Since Specht modules have basis elements indexed by standard Young tableaux, we are motivated to find a bijection between the sets B_1 and B_2 and certain sets of tableaux.
- A P_h-tableaux of shape λ is a filling of the diagram of λ with entries from P_h satisfying:
  - Each entry of P_h is used exactly once.
  - Rows are P_h-increasing.
  - Adjacent entries in columns are P_h-nondecreasing.
- P_h-tableaux were used by Gasharov to show that $X_{G_h}(x; q)$ is Schur-positive for any Hessenberg function h. [3]

Theorem 4. *If $h = (h(1), n, \ldots, n)$, then there is a bijection between the set of monomials in set B_1 and the set of P_h-tableaux with shape (1^n).*



Theorem 5. *If $h = (h(1), n, \ldots, n)$, then there is bijection between the set of monomials in set B_2 and the set of pairs of P_h-tableaux and standard tableaux, both with shape $(2, 1^{n-2})$.*



Positivity in the elementary basis

- Stanley [6] showed that graphs with independence number 1 or 2 are e-positive. When $h = (h(1), n, \ldots, n)$, the vertices $2, \ldots, n$ in G_h form a clique, so these graphs are in that category.
- We give another proof of this fact from the Hessenberg variety: Theorem 3 says that $H^*(\mathrm{Hess}(S, h))$, as an $\mathfrak{S}_n$-module, forms a permutation representation.
- The Frobenius character of a permutation representation is the complete homogeneous symmetric function h_λ, and applying the involution ω gives an e-positive expansion of $X_{G_h}(x; q)$ for this Hessenberg function.

References

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