On the f-vectors of poset associahedra

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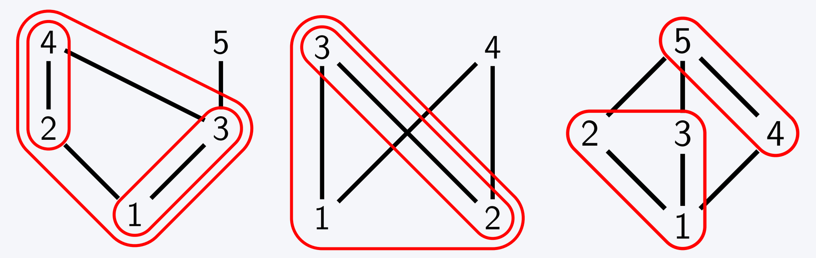
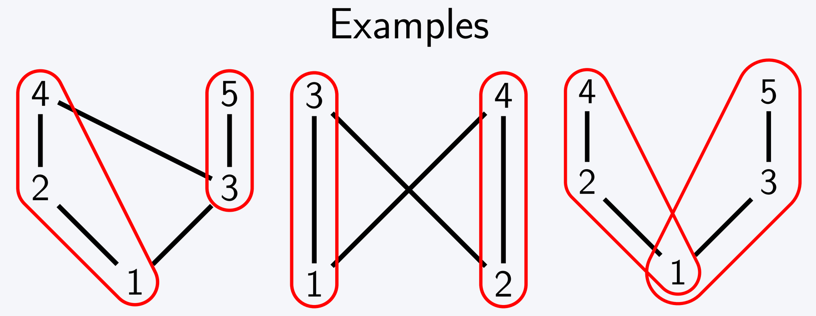
Background: Tubes and Tubings

- A *proper tube* of a poset P is a connected, convex subset $\tau \subset P$ such that $1 < |\tau| < |P|$.
- For disjoint tubes σ, τ we say $\tau \prec \sigma$ if there exists $a \in \tau, b \in \sigma$ such that $a \prec b$.
- A *proper tubing* T of P is a set of proper tubes of P such that any pair of tubes is nested or disjoint and that the relation $\prec$ is acyclic.



Examples



Non-examples

Background: Poset Associahedra

Theorem (Galashin) For a finite, connected poset P, the *poset associahedron* $\mathscr{A}(P)$ is a simple, convex polytope of dimension $|P|-2$ whose face lattice is isomorphic to the set of proper tubings ordered by reverse inclusion.

In 2023, Sack provided an explicit realization of poset associahedra. Independently, Mantovani, Padrol, and Pilaud provide a realization as a special case of acyclonestohedra.

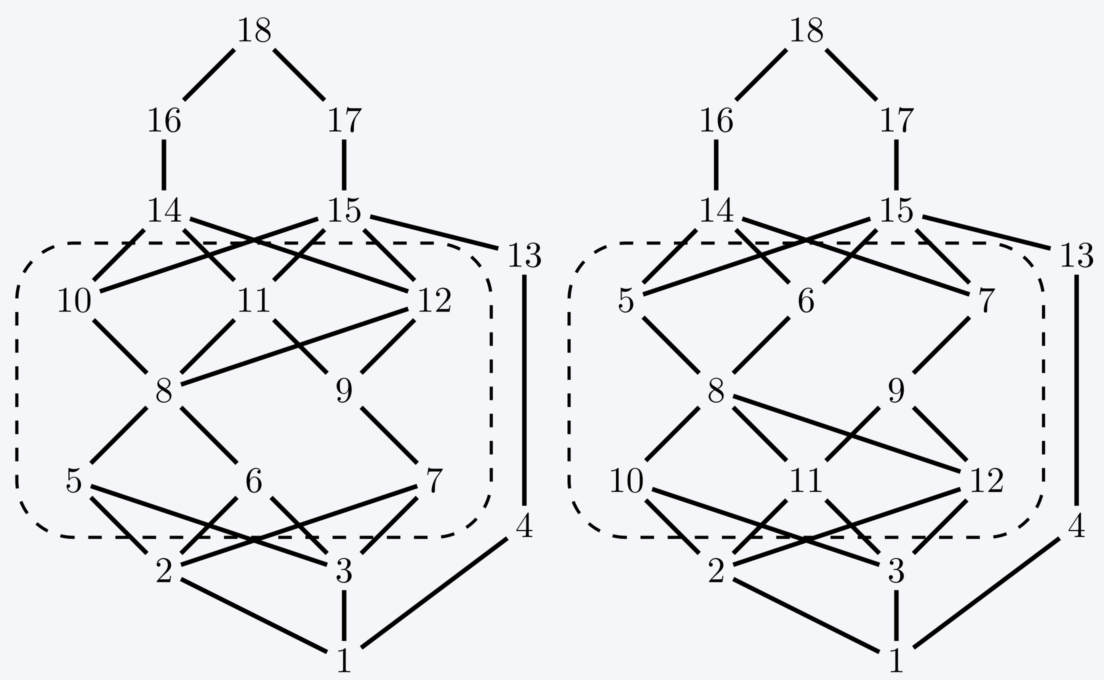
Background: Comparability Invariance

Let $(P, \preceq_P)$ be a poset. The comparability graph $G(P)$ has vertex set P where $\{x, y\}$ is an edge if $x \prec_P y$ or $y \prec_P x$. A property is said to be a *comparability invariant* if it only depends on $G(P)$.

$S \subseteq P$ is called *autonomous* if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

A *flip* of S is the replacement of S by S^{op}, that is S with all relations reversed.



An autonomous subset S of a poset P. A flip of S.

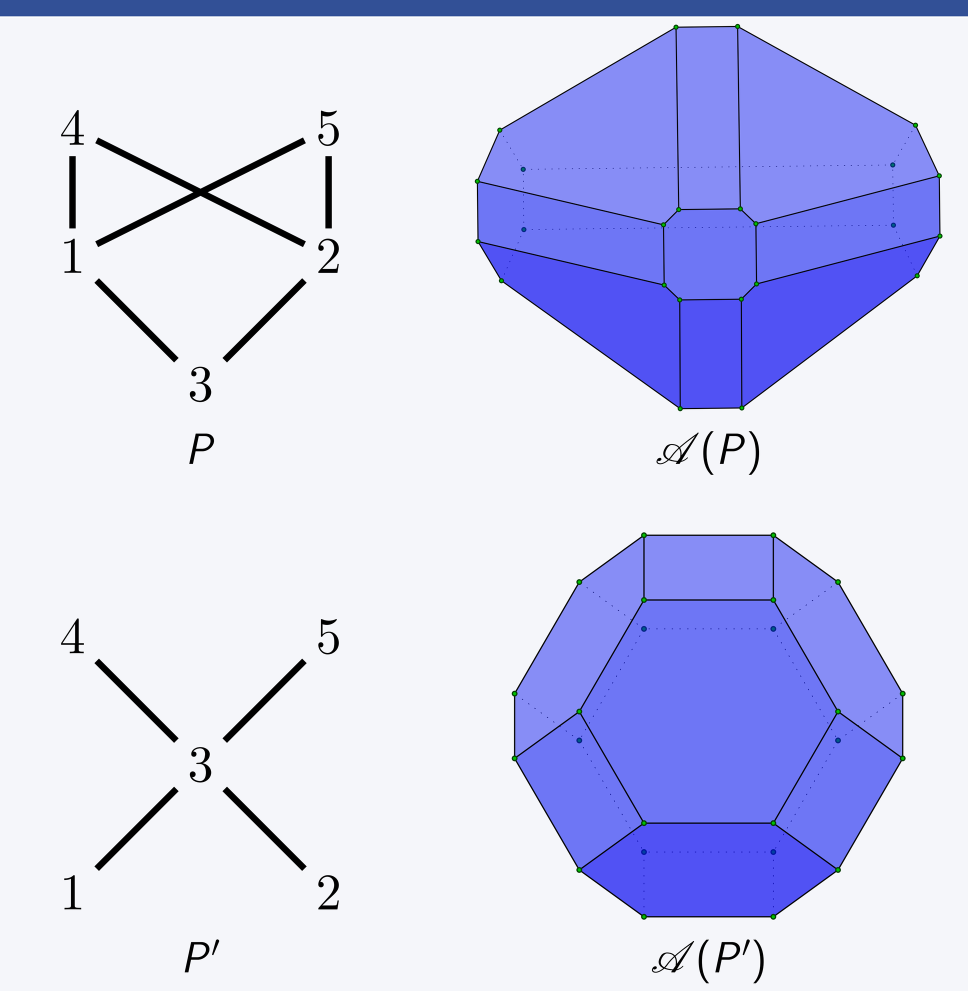
Main Result 1: Comparability Invariance

Theorem (Dreesen, Poguntke, Winkler (1985)) Two posets with the same comparability graph are connected by a sequence of flips of autonomous subsets.

Corollary A property is a comparability invariant if it is invariant under flips of autonomous subsets.

Theorem The f-vector of $\mathscr{A}(P)$ is a comparability invariant.

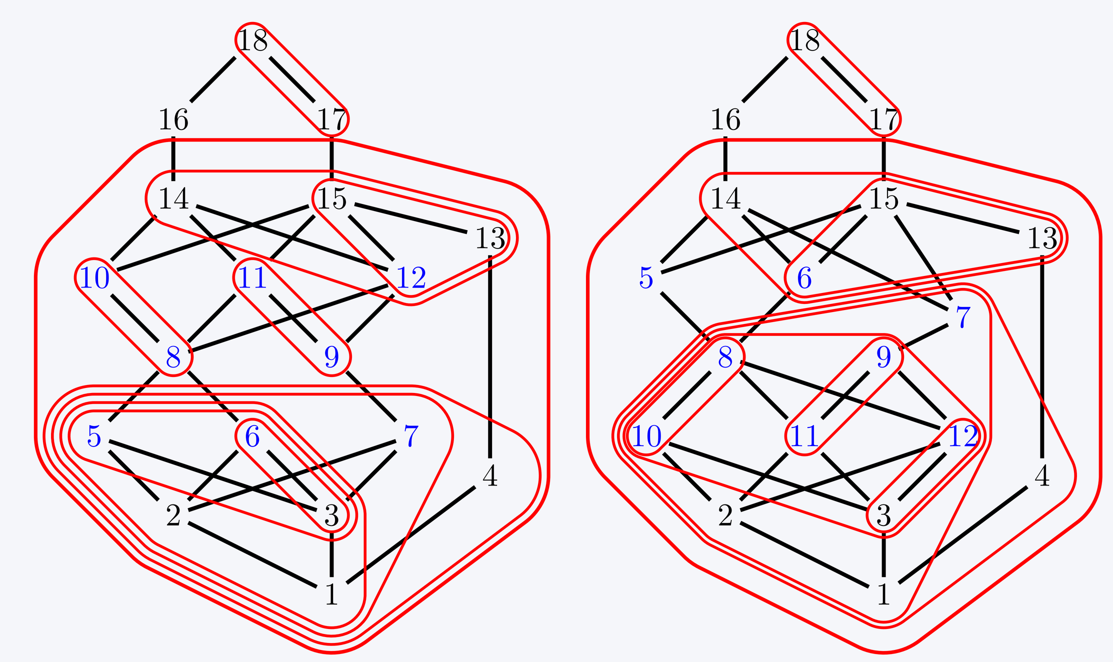
Example



Both $\mathscr{A}(P)$ and $\mathscr{A}(P')$ have f-vectors of $(24, 36, 14, 1)$.

Proof Idea

We define a map $\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$ on the level of tubings.



A proper tubing T on P. $\Phi_{P,S}(T)$

$$|\Phi_{P,S}(T)| = |T| \text{ and } \Phi_{P',S} \circ \Phi_{P,S} = \mathrm{id}.$$

Main Result 2

Brooms and Lollipops

We make the following definitions:

- The broom poset $A_{n,k}$ is the ordinal sum of a chain on $n+1$ elements and an antichain of k elements.
- The lollipop graph $L_{n,k}$ is a path graph on n vertices and a complete graph on k vertices joined by an edge.
- $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$

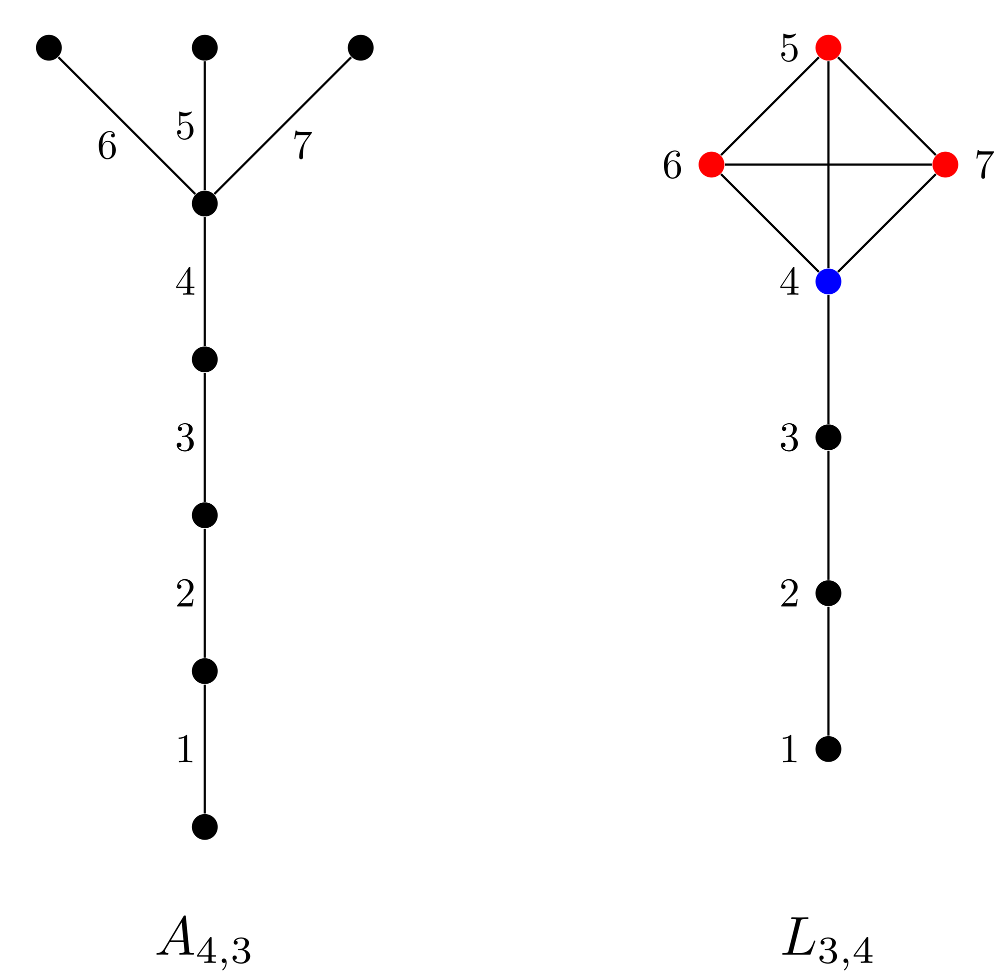
Laplante-Anfossi observed that the *poset associahedron* of a tree is combinatorially equivalent to the *graph associahedron* of the tree's line graph. In particular, $L_{n-1,k+1}$ is the line graph of $A_{n,k}$.

Theorem

Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents. Furthermore, $\mathscr{A}(A_{n,k})$ has

$$\frac{k+1}{n+k+1}\binom{2n+k}{n} \cdot k!$$

vertices.



The broom poset $A_{4,3}$ and lollipop graph $L_{3,4}$.

Background: Polyhedral Combinatorics

Definition For a d-dimensional polytope P, the f-vector is the sequence $(f_0, \ldots, f_d)$ where f_i is the number of faces of P of dimension i. The f-polynomial of P is

$$f(t) = \sum_{i=0}^{d} f_i t^i.$$

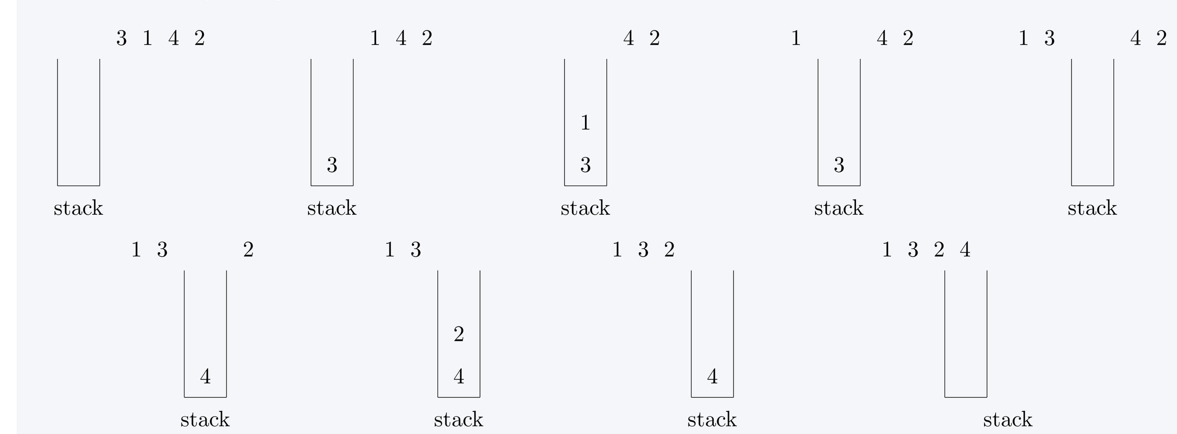
The h- and γ- polynomials are defined by

$$f_P(t) = h_P(t+1),$$
$$h_P(t) = (1+t)^d \gamma\left(\frac{t}{(1+t)^2}\right).$$

Background: Stack-sorting

West's stack-sorting map $s : \mathfrak{S}_n \to \mathfrak{S}_n$ is an algorithm that partially sorts a permutation using a stack. Given a permutation $\pi \in \mathfrak{S}_n$, $s(\pi)$ is obtained through the following procedure. Iterate through the entries of π. In each iteration,

- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- otherwise, pop the entry at the top of the stack to the end of the output permutation.



Example of $s(3142)$

Bonus Theorems

Theorem We obtain infinitely many polyhedra with f-vectors equal to permutohedra, but which are not combinatorially equivalent to permtuohedra.

Corollary For all n and k, the h-vector of $\mathscr{A}(A_{n,k})$ is γ-positive.

Theorem For all n, the h-polynomial of $\mathscr{A}(A_{n,2})$ is real-rooted.

Open Questions

1. Give a proof of comparability invariance of the f-vector of poset associahedra that is "direct".
2. Are the h-polynomials of all poset associahedron real-rooted or γ-positive?

References

[1] Bernhardine Dreesen, Werner Poguntke, and Peter Winkler. Comparability invariance of the fixed point property. *Order*, 2:269–274, 1985.

[2] Pavel Galashin. P-associahedra. *Selecta Mathematica*, 30(1):6, 2024.

[3] Chiara Mantovani, Arnau Padrol, and Vincent Pilaud. Acyclonestohedra. in prep.