Monomial expansions for q-Whittaker and modified Hall-Littlewood polynomials

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Introduction

Let λ be a partition. For $n \geq 1$, let X_n denote the tuple of indeterminates $x_1, x_2, \cdots, x_n$. The q-Whittaker polynomial $W_\lambda(X_n; q)$ and the modified Hall-Littlewood polynomial $Q'_\lambda(X_n; q)$ are well-studied specializations of the modified Macdonald polynomial. In this article, our focus will be on three monomial expansions: the so-called *fermionic formulas* [4, (0.2), (0.3)] and the inv- and quinv-expansions arising from specializations of the formulas of Haglund-Haiman-Loehr [2] and Ayyer-Mandelshtam-Martin [1].

- Given λ, let $\mathsf{dg}(\lambda)$ be its Young diagram. Fix $n \geq 1$, and let $\mathcal{F}(\lambda)$ denote the set of all maps ("fillings") $F : \mathsf{dg}(\lambda) \to [n]$ where $[n] = \{1, 2, \cdots, n\}$
- For $\lambda = (6, 4, 2)$, and $n = 8$ following is a filling of $\mathsf{dg}(\lambda)$

$F =$

1	3	5	5	6	8
2	4	2	7		
3	1				

Following Haglund-Haiman-Loehr [2] and Ayyer-Mandelshtam-Martin [1], there are statistics *inv, quinv* and *maj* on $\mathcal{F}(\lambda)$ such that

$$\widetilde{H}_\lambda(X_n; q, t) = \sum_{F \in \mathcal{F}(\lambda)} x^F q^{v(F)} t^{\mathsf{maj}(F)} \tag{1}$$

where $v \in \{\mathsf{inv}, \mathsf{quinv}\}$. Expand $\widetilde{H}_\lambda(X_n; q, t)$ in powers of t; our interest lies in the coefficients of the lowest and highest powers [8, (3.1)]:

$$\widetilde{H}_\lambda(X_n; q, t) = \mathcal{H}_\lambda(X_n; q)t^0 + \cdots + W_\lambda(X_n; q)t^{\eta(\lambda)} \tag{2}$$

Column strict filling (CSF)

- A filling F is a *column strict*, if the values of F strictly increase down each column. Let $\mathrm{CSF}(\lambda)$ denotes the collection of all column strict fillings of shape λ.
- Let $F \in \mathcal{F}(\lambda)$. Then $\mathsf{maj}(F) = \eta(\lambda)$ iff $F \in \mathrm{CSF}(\lambda)$.
- From (1), we obtain for $v \in \{\mathsf{inv}, \mathsf{quinv}\}$:

$$W_\lambda(X_n; q) = \sum_{F \in \mathrm{CSF}(\lambda)} x^F q^{v(F)} \tag{3}$$

Partition overlaid pattern

- Let $n \geq 1$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n nonzero parts. Let $\mathrm{GT}(\lambda)$ denote the set of integral Gelfand-Tsetlin (GT) patterns with bounding row λ. Given $T \in \mathrm{GT}(\lambda)$, we denote its entries by T_i^j for $1 \leq i \leq j \leq n$ as in Figure 1.
- Define the North-East and South-East differences of T by: $\mathsf{NE}_{ij}(T) = T_i^{j+1} - T_i^j$ and $\mathsf{SE}_{ij}(T) = T_i^j - T_{i+1}^{j+1}$ for $1 \leq i \leq (j+1) \leq n$.
- A *partition overlaid pattern* (POP) of shape λ is a pair (T, Λ), where $T \in \mathrm{GT}(\lambda)$ and $\Lambda = (\Lambda_{ij} : 1 \leq i \leq j < n)$ is a tuple of partitions such that each Λ_{ij} fits into a rectangle of size $NE_{ij}(T) \times SE_{ij}(T)$. Let $\mathrm{POP}(\lambda)$ denotes the set of all partition overlaid patterns of shape λ.
- The q-Whittaker polynomial can be expressed as

$$W_\lambda(X_n; q) = \sum_{(T,\Lambda) \in \mathrm{POP}(\lambda)} x^T q^{|\Lambda|} \tag{4}$$

where $|\Lambda| = \sum_{i,j} |\Lambda_{ij}|$.

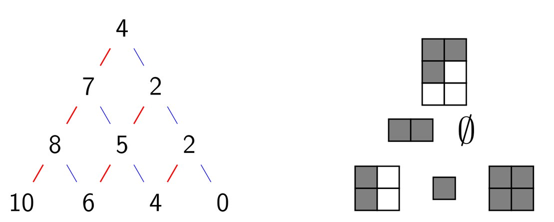


Figure 1: A GT-pattern for $n = 4$. On the right is a partition overlay compatible with this GT-pattern.

Projection and Branching for Partition overlaid patterns

- Given λ, we say that $\mu = (\mu_1, \mu_2, \cdots, \mu_{n-1})$ *interlaces* λ (and write $\mu \prec \lambda$) if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $1 \leq i < n$.
- The q-Whittaker polynomials have the following important properties which readily follow from (4)
  - i (*projection*) $W_\lambda(X_n; q = 0) = s_\lambda(X_n)$, the Schur polynomial.
  - ii (*branching*) $W_\lambda(x_1, x_2, \cdots, x_{n-1}, x_n = 1; q) = \sum_{\mu \prec \lambda} \prod_{1 \leq i < n} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}_q \cdot W_\mu(X_{n-1}; q)$
- The combinatorial shadow of projection is the map $\mathsf{pr} : \mathrm{POP}(\lambda) \to \mathrm{GT}(\lambda)$ given by $\mathsf{pr}(T, \Lambda) = T$.
- Define *combinatorial branching* to be the map $\mathsf{br} : \mathrm{POP}(\lambda) \to \coprod_{\mu \prec \lambda} \mathrm{POP}(\mu)$ defined by $\mathsf{br}(T, \Lambda) = (T^\dagger, \Lambda^\dagger)$ where $T^\dagger$ is obtained from T by deleting its bottom row, and $\Lambda^\dagger$ is obtained from Λ by deleting the overlays Λ_{ij} with $j = n - 1$.
- For $(T, \Lambda) \in \mathrm{POP}(\lambda)$, define $\mathsf{boxcomp}(T, \Lambda) = (T, \Lambda^c)$ where for each i, j, $(\Lambda^c)_{ij}$ is defined to be the complement of Λ_{ij} in its bounding rectangle of size $\mathsf{NE}_{ij}(T) \times \mathsf{SE}_{ij}(T)$.

Projection and branching for Column strict fillings

- Given $F \in \mathrm{CSF}(\lambda)$, let $\mathsf{rsort}(F) \in \mathrm{SSYT}(\lambda) \cong \mathrm{GT}(\lambda)$, denote the filling obtained from F by sorting entries of each row in ascending order. we think of rsort as the projection map in the CSF setting.
- Let $\ell \geq 1$ and suppose $\sigma = (\sigma_1 < \sigma_2 < \cdots < \sigma_{\ell-1})$ and $\tau = (\tau_1 < \tau_2 < \cdots < \tau_\ell)$ are column tuples of length $\ell - 1$ and ℓ respectively. We set $\sigma_0 = 0$ and let k denote the maximum element of the (non-empty) set $\{1 \leq i \leq \ell : \sigma_{i-1} < \tau_i\}$.
- Define $\mathsf{splice}(\sigma, \tau) = (\overline{\sigma}, \overline{\tau})$ where

$$\overline{\sigma}_i = \begin{cases} \sigma_i & 1 \leq i < k \\ \tau_i & k \leq i \leq \ell \end{cases} \quad \text{and} \quad \overline{\tau}_i = \begin{cases} \tau_i & 1 \leq i < k \\ \sigma_i & k \leq i < \ell \end{cases}$$

For instance when $(\sigma, \tau) = \left(\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix} \right)$, we get $(\overline{\sigma}, \overline{\tau}) = \left(\begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \right)$.

- The delete-and-splice rectification ("dsplice") map on $F \in \mathrm{CSF}(\lambda)$ is defined as follows:
  - i delete all cells in F containing the entry n and let $F^\dagger$ denote the resulting filling. While its column entries remain strictly increasing, $F^\dagger$ may no longer be of partition shape.
  - ii Applying splice map between appropriate columns $F^\dagger$ we can produces a CSF of partition shape (filled by numbers between 1 and $n-1$), which we denote $\mathsf{dsplice}(F)$. The following properties hold:

Proposition 1

With notation as above: (i) $D := \mathsf{dsplice}(F)$ is independent of the intermediate choices of columns in step 2 of the procedure. (ii) $\mathsf{rsort}(D)$ is obtained from $\mathsf{rsort}(F)$ by deleting the cells containing the entry n. (iii) If μ and λ are the shapes of D and F respectively, then $\mu \prec \lambda$.

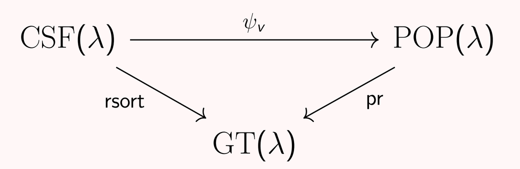
We consider $\mathsf{dsplice}$ to be the combinatorial branching map in the CSF context.

Main Theorem

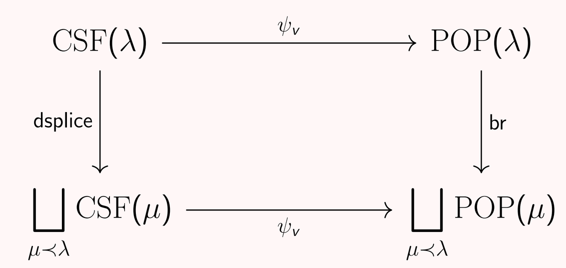
For any $n \geq 1$ and any partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ with at most n nonzero parts, there exist two bijections ψ_{inv} and ψ_{quinv} from $\mathrm{CSF}(\lambda)$ to $\mathrm{POP}(\lambda)$ with the following properties:

1. If $\psi_v(F) = (T, \Lambda)$, then $x^F = x^T$ and $v(F) = |\Lambda|$, for $v = \mathsf{inv}$ or quinv.
2. The following diagrams commute ($v = \mathsf{inv}$ or quinv):

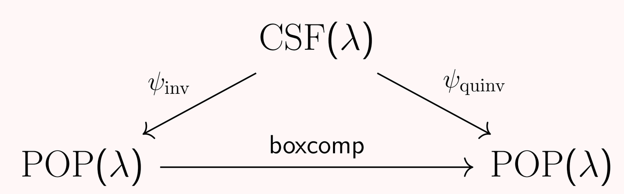
(A)



(B)



3. The two bijections are related via the commutative diagram:



Proof sketch

- For a partition λ, the augmented diagram $\widehat{\mathsf{dg}}(\lambda)$ is $\mathsf{dg}(\lambda)$ together with one additional cell below the last cell in each column.
- Given $F \in \mathrm{CSF}(\lambda)$, a *quinv-triple* (resp. *refinv-triple*) in F is a triple of cells (x, y, z) in $\widehat{\mathsf{dg}}(\lambda)$ such that (i) $x, z \in \mathsf{dg}(\lambda)$ and z is to the right (resp. to the left) of x in the same row, (ii) y is the cell immediately below x in its column, (iii) $F(x) < F(z) < F(y)$, where we set $F(y) = \infty$ if y lies outside $\mathsf{dg}(\lambda)$.
- The quinv-triples considered in [1] for $F \in \mathcal{F}(\lambda)$ reduce to this description when F is a CSF rather than a general filling.

Proposition 2

For $F \in \mathrm{CSF}(\lambda)$, $\mathsf{inv}(F)$ equals the number of refinv-triples of F.

Definition of ψ_{quinv}

- Given $F \in \mathrm{CSF}(\lambda)$, for each cell $c \in \mathsf{dg}(\lambda)$, let $\mathsf{zcount}(c, F)$ = the number of quinv-triples (x, y, z) in F with $z = c$. Clearly

$$\sum_{c \in \mathsf{dg}(\lambda)} \mathsf{zcount}(c, F) = \mathsf{quinv}(F). \tag{5}$$

- Let $\mathsf{cells}(i, j, F) = \{c \in \mathsf{dg}(\lambda) : c \text{ is in the } i^{th} \text{ row and } F(c) = j + 1\}$ for $1 \leq i \leq j + 1 \leq n$.
- It follows that $|\mathsf{cells}(i, j, F)| = NE_{ij}(T)$, where $T = \mathsf{rsort}(F)$.
- If $c \in \mathsf{cells}(i, j, F)$, then $\mathsf{zcount}(c, F) \leq \mathsf{SE}_{ij}(T)$.
- If $c, d \in \mathsf{cells}(i, j, F)$ with c lying to the right of d, then $\mathsf{zcount}(c, F) \geq \mathsf{zcount}(d, F)$.

Let $F \in \mathrm{CSF}(\lambda)$ and $T = \mathsf{rsort}(F)$. For each $1 \leq i \leq j + 1 \leq n$, consider the sequence

$$\Lambda_{ij} = (\mathsf{zcount}(c, F) : \ c \in \mathsf{cells}(i, j, F) \text{ traversed right to left in row } i). \tag{6}$$

Define

$$\psi_{quinv}(F) = (T, \Lambda),$$

where $\Lambda = (\Lambda_{ij} : 1 \leq i \leq j < n)$, then $(T, \Lambda) \in \mathrm{POP}(\lambda)$. Clearly, $x^F = x^T$ and (5) implies $\mathsf{quinv}(F) = |\Lambda|$.

Example

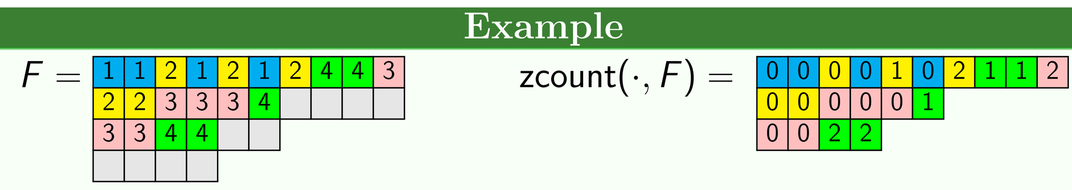


Figure 2: Here $F \in \mathrm{CSF}(\lambda)$ for $\lambda = (10, 6, 4, 0)$ and $n = 4$. Cells of F are coloured according to their entries. The gray cells are the extra cells in the augmented diagram $\widehat{\mathsf{dg}}(\lambda)$. On the right are cellwise zcount values. Here $\mathsf{quinv}(F) = 12$.

Definition of ψ_{inv}

Given $F \in \mathrm{CSF}(\lambda)$ and $c \in \mathsf{dg}(\lambda)$, define $\overline{\mathsf{zcount}}(c, F)$ = the number of refinv-triples (x, y, z) in F with $z = c$. In light of Proposition 2, it is clear that

$$\sum_{c \in \mathsf{dg}(\lambda)} \overline{\mathsf{zcount}}(c, F) = \mathsf{inv}(F) \tag{7}$$

We have the following relation between $\overline{\mathsf{zcount}}$ and zcount:

Proposition 3

Let $F \in \mathrm{CSF}(\lambda)$ and $T = \mathsf{rsort}(F)$. Let $1 \leq i \leq j + 1 \leq n$ and $c \in \mathsf{cells}(i, j, F)$. Then $\mathsf{zcount}(c, F) + \overline{\mathsf{zcount}}(c, F) = \mathsf{SE}_{ij}(T)$.

Given $F \in \mathrm{CSF}(\lambda)$, let $T = \mathsf{rsort}(F)$. For each $1 \leq i \leq j < n$, consider the sequence:

$$\overline{\Lambda}_{ij} = (\overline{\mathsf{zcount}}(c, F) : \ c \in \mathsf{cells}(i, j, F) \text{ traversed left to right in row } i)$$

It follows from Proposition 3 that $\overline{\Lambda}_{ij}$ is the box-complement of Λ_{ij} in the $\mathsf{NE}_{ij}(T) \times \mathsf{SE}_{ij}(T)$ rectangle. Let $\overline{\Lambda} = (\overline{\Lambda}_{ij} : 1 \leq i \leq j < n)$, we define $\psi_{\mathsf{inv}}(F) = (T, \overline{\Lambda})$. We have $x^F = x^T$, and $\mathsf{inv}(F) = |\overline{\Lambda}|$.

Construction of ψ^{-1}_{inv} and $\psi^{-1}_{\mathsf{quinv}}$

Given $(T, \Lambda) \in \mathrm{POP}(\lambda)$, construct the filling $F := \psi^{-1}_{\mathsf{inv}}(T, \Lambda) \in \mathrm{CSF}(\lambda)$ inductively row-by-row, from the bottom (n^{th}) row to the top as follows:

(a) fill all cells of the n^{th} row (if nonempty) with n,

(b) let $1 \leq i \leq j < n$; assuming that all rows of F strictly below row i have been completely determined and that the locations of entries $> (j + 1)$ in row i have been determined, we now need to fill $\mathsf{NE}_{ij}(T)$ many cells of row i with the entry $j + 1$. It turns out that the number of cells in row i in which we can potentially put a $j + 1$ without violating the CSF condition thus far is exactly $k + \ell$ where $k = \mathsf{NE}_{ij}(T)$ and $\ell = \mathsf{SE}_{ij}(T)$. We label these cells $0, 1, \cdots, k + \ell - 1$ from right to left (left-to-right when defining $\psi^{-1}_{\mathsf{quinv}}$). The partition Λ_{ij} can be viewed as a k-tuple of candidate cells in row i; we put the entry $j + 1$ into these.

(c) fill the remaining cells of row i with the entry i.

Example

Let $n = 4$, $\lambda = (10, 6, 4, 0)$ and let T, Λ be the GT pattern and overlay depicted in Figure 1. Then $\psi^{-1}_{\mathsf{quinv}}(T, \Lambda)$ is precisely the CSF F of Figure 2, while

$\psi^{-1}_{\mathsf{inv}}(T, \Lambda) =$

2	1	1	1	3	2	1	4	4	2
3	3	2	2	4	3				
4	4	3	3						

Local Weyl modules and limit constructions

Using Theorem 1 to replace POPs with CSFs as our model in [7, Corollary 5.13], we deduce :

Proposition 4

Fix $n \geq 2$ and consider the partition $\theta = (2, 1, 1, \cdots, 1, 0)$ with $n - 1$ nonzero parts and $|\theta| = n$. For $k \geq 0$, let $\mathcal{C}_k$ denote the set of CSFs F of shape $k\theta$ and entries in $[n]$, with the property that either 1 occurs in the first column of F or 1 does not occur in its last column. Then $\sum_{k \geq 0} \sum_{F \in \mathcal{C}_k} x^F q^{k^2 - \mathsf{inv}(F)}$ equals the character of $L(\Lambda_0)$.

Concluding Remarks

- For the modified Hall-Littlewood polynomials $Q'_{\lambda'}(X_n; q)$ of (3), the fermionic formula appears in [4, (0.2)]. Analogous to (4), this can now be recast as a *weighted sum* over *partition overlaid plane-partitions* (POPP) of shape λ. Min Theorem takes the form of bijections from $\mathrm{WDF}(\lambda)$ to $\mathrm{POPP}(\lambda)$ (or equivalently, from tabloids to partition overlaid reverse-plane-partitions).
- The bijections of Theorem 1 (and those indicated above for the modified Hall-Littlewood case) have an attractive interpretation in terms of lattice-path diagrams.

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