Pattern-avoiding polytopes and Cambrian lattices

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OEIS A003121

1. # of shifted standard tableaux of staircase shape
2. # longest chains in the Tamari lattice
3. # of reduced words of a certain commutation class of the long permutation
4. # of linear extensions of the poset of join irreducibles of the 132 and 312 avoiding permutations
5. the normalized volume of a subpolytope of the Birkhoff polytope whose vertices are the permutation matrices of 132 and 312 avoiding permutations

Davis and Sagan's Question [1]

Is the above polytope in 5 unimodularly equivalent to the order polytope of the poset in 4?

The c-Birkhoff Polytope Birk(c)

A Coxeter element in S_{n+1} is a product of all simple reflections $s_1, \ldots, s_n$. Ex. $s_2s_3s_1s_4 \in S_5$

If $c = s_{i_1}s_{i_2}\ldots s_{i_\ell}$, then we define the expression $c^\infty := i_1 \cdots i_\ell \mid i_1 \cdots i_\ell \mid i_1 \cdots i_\ell \mid \cdots$

The c-sorting word of w, denoted $\mathsf{sort}_c(w)$, is the lexicographically left-most reduced expression for w which appears as a subword of c^∞.

Ex. Let $c = s_1s_2s_3 \in S_4$, $w_0 = s_1s_2s_1s_3s_2s_1 = s_1s_2s_3s_1s_2s_1$, ①②③ | ①②3 | ①23 | 123 | …

So $\mathsf{sort}_c(w_0) = s_1s_2s_3s_1s_2s_1$.

w is a c-singleton iff the c-sorting word of w is a prefix of the c-sorting word of the long element w_0 up to commutations.

Ex. The c-singletons for $c = s_1s_2s_3$ (Tamari) are

$$\mathrm{Id}, s_1, s_1s_2, s_1s_2s_1, s_1s_2s_3s_1, s_1s_2s_3s_1s_2, s_1s_2s_3s_1s_2s_1, s_1s_2s_3.$$

The c-Birkhoff polytope, denoted Birk(c), is the convex hull of $\{X_w \mid w \text{ is a } c\text{-singleton}\}$ where X_w is the permutation matrix for w.

Ex. Birk$(s_1s_2s_3)$ is the convex hull of the 8 points in $\mathbb{R}^{16}$:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}.$$

The Order Polytope of a Heap $\mathcal{O}(\mathcal{H}_c)$

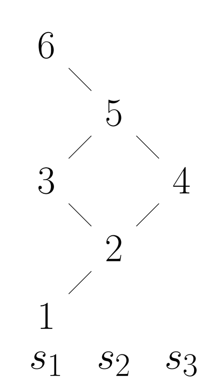
The heap of a reduced word $q = s_{i_1}\cdots s_{i_k}$ for w is a poset on $\{1, \ldots, k\}$ with cover relations

$a \prec b$ if both $a < b$ and the corresponding simple generators s_{i_a}, s_{i_b} do not commute.

Denote the poset $\mathcal{H}_c := \mathsf{Heap}(\mathsf{sort}_c(w_0))$ for a Coxeter element c.

The order polytope of a poset is the convex hull of the indicator vectors of its order ideals.

Ex. The vertices of $\mathcal{O}(\mathcal{H}_{s_1s_2s_3})$ are the 8 points in $\mathbb{R}^6$ (columns of the matrix below):



$$\begin{bmatrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&1&1&0\\0&0&0&0&1&1&1&1\\0&0&0&1&1&1&1&0\\0&0&1&1&1&1&1&1\\0&1&1&1&1&1&1&1\end{bmatrix}$$

The dimension of an order polytope is the # of vertices in the poset (full-dimensional); the normalized volume is # of linear extensions of the poset.

Constructions

$$\underset{\text{Birk}(c)}{\mathbb{R}^{(n+1)^2}} \xrightarrow[\text{lattice-preserving projection}]{\Pi_c} \mathbb{R}^{\binom{n+1}{2}} \xrightarrow[\text{unimodular transformation}]{\mathcal{U}_c} \underset{\mathcal{O}(\mathcal{H}_c)}{\mathbb{R}^{\binom{n+1}{2}}}$$

Theorem

For any Coxeter element c, the two polytopes Birk(c) and $\mathcal{O}(\mathcal{H}_c)$ are unimodularly equivalent.

A Tamari Example $(c = s_1s_2s_3)$

The unimodular transformation from Birk(c) to $\mathcal{O}(\mathcal{H}_c)$ is below:

$$\begin{bmatrix}1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&0&0&1&0&0\\0&0&0&1&1&0\\1&0&0&1&0&1\end{bmatrix} \left[\begin{array}{ccccccc|c}0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&1\\0&0&0&1&1&1&0&0\\0&0&1&0&0&0&1&1\\0&1&1&0&0&0&0&1\end{array}\right] = \left[\begin{array}{ccccccc|c}0&0&0&0&0&0&1&0\\0&0&0&0&0&1&1&0\\0&0&0&0&1&1&1&1\\0&0&0&1&1&1&1&0\\0&0&1&1&1&1&1&1\\0&1&1&1&1&1&1&1\end{array}\right].$$

Left matrix: $\mathcal{U}_c$, the unimodular transformation matrix;

Middle matrix: columns are the projection of the vertices of Birk(c) under the projection

$$\Pi_c = \begin{bmatrix} &6&4&1\\ & &5&2\\ & & &3\\ &X&X& \end{bmatrix};$$

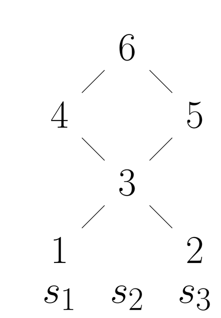
Right matrix: columns are the vertices of $\mathcal{O}(\mathcal{H}_c)$.

Note the last column of the middle (resp. right) matrix is 5$^{\text{th}}$ column - 4$^{\text{th}}$ column + 3$^{\text{rd}}$ column, it suffices to solve the system of linear equations before |.

A Bipartite Example $(c = s_1s_3s_2)$

$\mathsf{sort}_c(w_0) = s_1s_3s_2s_1s_3s_2$ and Birk(c) is the convex hull of the following 9 points in $\mathbb{R}^{16}$:

$$\begin{array}{ccccc}\text{Id} & s_1 & s_1s_3 & s_1s_3s_2 & s_1s_3s_2s_1\\ \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} & \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} & \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} & \begin{bmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{bmatrix} & \begin{bmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}\\ \begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix} & \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} & \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} & \begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix} & \begin{bmatrix} &6&X&3\\ & &1&4\\ & & &5\\2&X& & \end{bmatrix}\\ s_1s_3s_2s_1s_3 & s_1s_3s_2s_1s_3s_2 & s_3 & s_1s_3s_2s_3 & \Pi_c\end{array}$$



Then the unimodular transformation from Birk(c) to $\mathcal{O}(\mathcal{H}_c)$ can be written as the following:

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&1&1&1&0\\0&0&1&0&0&1\end{bmatrix} \left[\begin{array}{ccccccc|cc}0&0&0&0&0&0&1&0&0\\0&0&0&0&0&1&1&0&1\\0&0&0&0&1&1&1&0&0\\0&0&0&1&0&0&0&0&1\\0&0&1&0&0&0&0&1&0\\0&1&1&1&0&0&0&0&1\end{array}\right] = \left[\begin{array}{ccccccc|cc}0&0&0&0&0&0&1&0&0\\0&0&0&0&0&1&1&0&1\\0&0&0&0&1&1&1&0&0\\0&0&0&1&1&1&1&0&1\\0&0&1&1&1&1&1&1&1\\0&1&1&1&1&1&1&0&1\end{array}\right].$$

Corollary

The normalized volume of the c-Birkhoff polyope is equal to the number of longest chains in the c-Cambrian lattice.

Notation

For a Coxeter element $c \in S_{n+1}$, we partition the integers in $[2, n]$ into lower-barred and upper-barred numbers $d_1 < d_2 < \cdots < d_r$ and $u_1 < u_2 < \cdots < u_s$, respectively. If s_{i-1} is to the left of s_i in c, i is a lower-barred number; otherwise, i is an upper-barred number.

c can be presented in cycle notation

$$c = (1\ \underline{d_1}\ \ldots\ \underline{d_r}\ (n+1)\ \overline{u_s}\ \ldots\ \overline{u_1}) = (\overline{u_s}\ \ldots\ \overline{u_1}\ 1\ \underline{d_1}\ \ldots\ \underline{d_r}\ (n+1)).$$

Ex. Let $c = s_1s_4s_3s_2s_6s_5s_7$, then $d_1, d_2, d_3 = \underline{2}, \underline{5}, \underline{7}$ and $u_1, u_2, u_3 = \overline{3}, \overline{4}, \overline{6}$. We get $c = (12578643)$.

Properties of c-singletons

Any c-singleton w, with permutation matrix $X_w = (X(i,j))_{i,j}$, satisfies the following properties.

Proposition (Positions of zeros) Given an upper-barred number u, if $1 \le j \le \min(u-1, n+1-u)$, then we cannot have $w(j) = u$. Given a lower-barred number d, if $\max(d+1, n+3-d) \le j \le n+1$, then we cannot have $w(j) = d$.

Proposition (Adding to one) For each $1 \le i \le \frac{n-1}{2}$ and $i+1 \le u \le n-i$, there exists a sequence $i = v_0 < v_1 < \cdots < v_d$, where $d \ge 1$, such that

$$\sum_{j=0}^{d}\sum_{i=1}^{u} X(i, v_j)$$

is equal to either 1 or d (depending on i and u).

Corollary (Tamari case) If $c = s_1s_2\cdots s_n$ and w is a c-singleton, then, for each $y \in [n+1]$ and $0 \le z \le y-1$, there is exactly one value in $\{w(1), \ldots, w(y)\}$ which is equivalent to z modulo y.

A Lattice-preserving Projection Π_c

To define Π_c, the first entries we will read are

$$\begin{aligned}&(d_1-1, d_1), (d_1-2, d_1), \ldots, (1, d_1),\\ &\ldots\\ &(d_r-1, d_r), (d_r-2, d_r), \ldots, (1, d_r),\\ &(n, n+1), (n-1, n+1), \ldots, (1, n+1).\end{aligned}$$

The remaining entries come from $u_s, \ldots, u_1$. For each u, take $u-1$ entries as follows:

- Let $m = \min(u-1, n+1-u)$.
- First take the m entries $(n+1, c^1(u), (n, c^2(u)), \ldots, (n+2-m, c^m(u))$.
- Then take the additional $u-1-m$ entries $(u-1, u), (u-2, u), \ldots, (m+1, u)$.

Ex. For $c = s_1s_4s_3s_2s_6s_5s_7$, we have $c = (12578643)$. We compute the projection Π_c below.

	28	X	X	24	X	18	11
		X	X	25	X	19	12
			X	26	6	20	13
				27	7	21	14
					8	22	15
	3			X		23	16
4	1	9		X			17
2	X	5	10	X		X	

0	①	0	0	⓪	0	⓪	⓪
0	0	0	0	①	0	⓪	⓪
1	0	0	0	⓪	⓪	⓪	⓪
0	0	1	0	⓪	⓪	⓪	⓪
0	0	0	1	0	⓪	⓪	⓪
0	⓪	0	0	0	1	0	⓪
⓪	⓪	⓪	0	0	0	1	⓪
⓪	0	⓪	⓪	0	0	0	1

Figure 1. Left: The projection Π_c. Right: permutation matrix for $s_1s_4s_3s_2$.

$$\Pi_c(s_1s_4s_3s_2) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1), \text{ and}$$
$$o(s_1s_4s_3s_2) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1).$$

Theorem Fix a Coxeter element c in S_{n+1}. There exists a $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ with 1's on the main diagonal such that $\mathcal{U}_c \circ \Pi_c(b_i) = o(b_i)$ for all $1 \leqslant i \leqslant \binom{n+1}{2}$. Furthermore, we have $\mathcal{U}_c \circ \Pi_c(w) = o(w)$ for any c-singleton w.

References

[1] Robert Davis and Bruce Sagan. Pattern-avoiding polytopes. *European J. Combin.*, 74:48–84, 2018.