

FRAMED POLYTOPES AND HIGHER CATEGORIES

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CELLULAR STRINGS ON FRAMED POLYTOPES

FRAMED POLYTOPES

Frame: ordered basis (e_1, \dots, e_d)

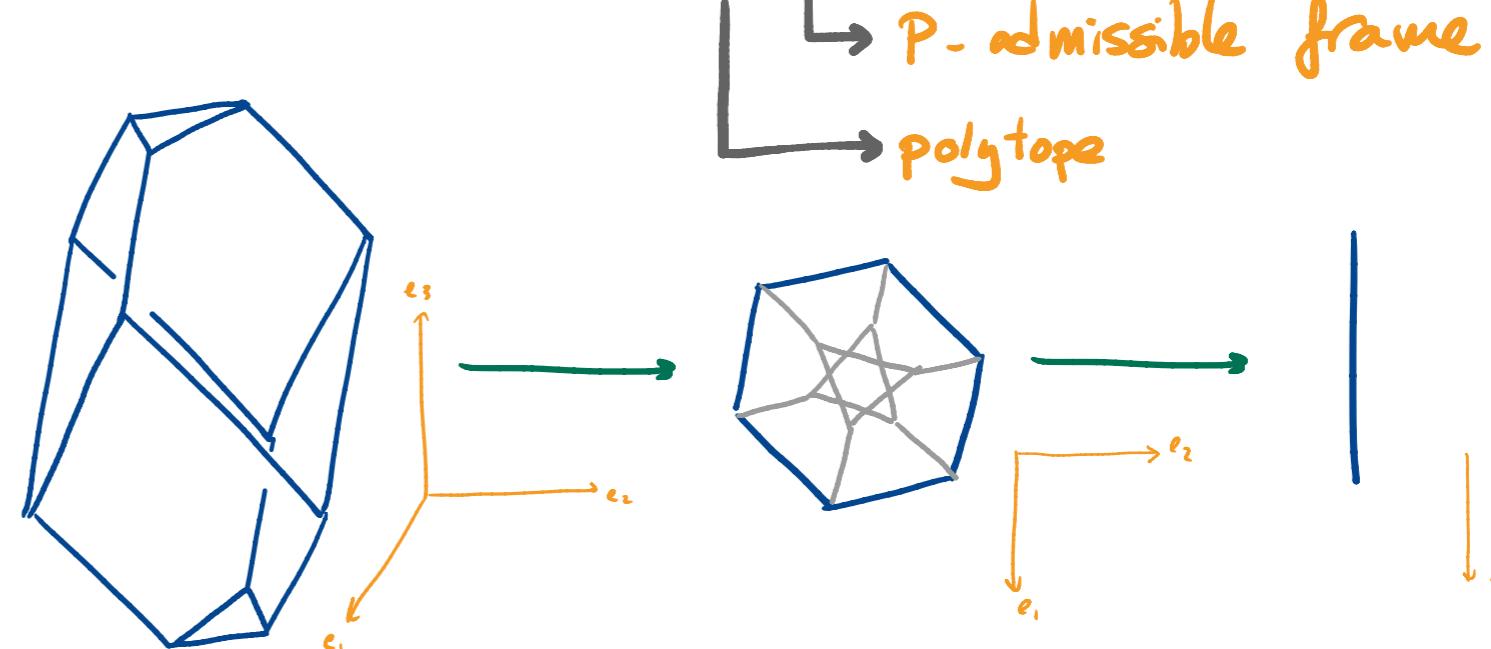
System of projections:

$$\mathbb{R}^d \xrightarrow{\pi_{d-1}} \mathbb{R}^{d-1} \xrightarrow{\pi_{d-2}} \mathbb{R}^{d-2} \rightarrow \dots \rightarrow \mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}$$

$\pi_k = \pi_{k-1}^{k-1} \circ \pi_{k-2}^{k-2} \circ \dots \circ \pi_1^1 : \mathbb{R}^d \rightarrow \mathbb{R}^k$ forgetting the last $d-k$ coordinates

P -admissible: $F \cong \pi_k(F)$ $\forall k$ -face F of P generic w.r.t. P

Framed polytope: (P, F)

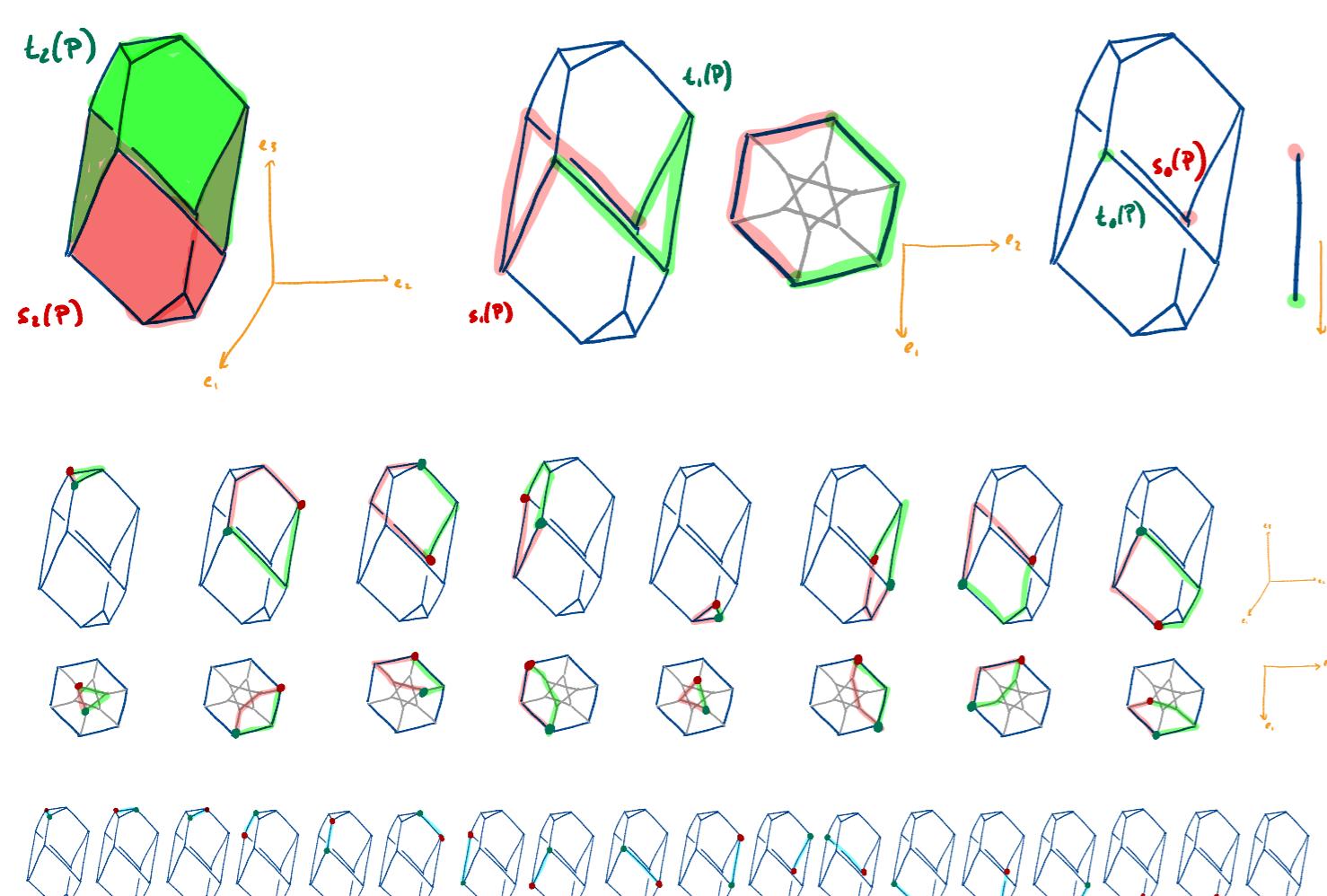


GLOBULAR STRUCTURES

$F \in P$ i -face $k \leq i$

k -source: $s_k(F) = k$ -faces in lower envelope of $\pi_{k+1}(F)$

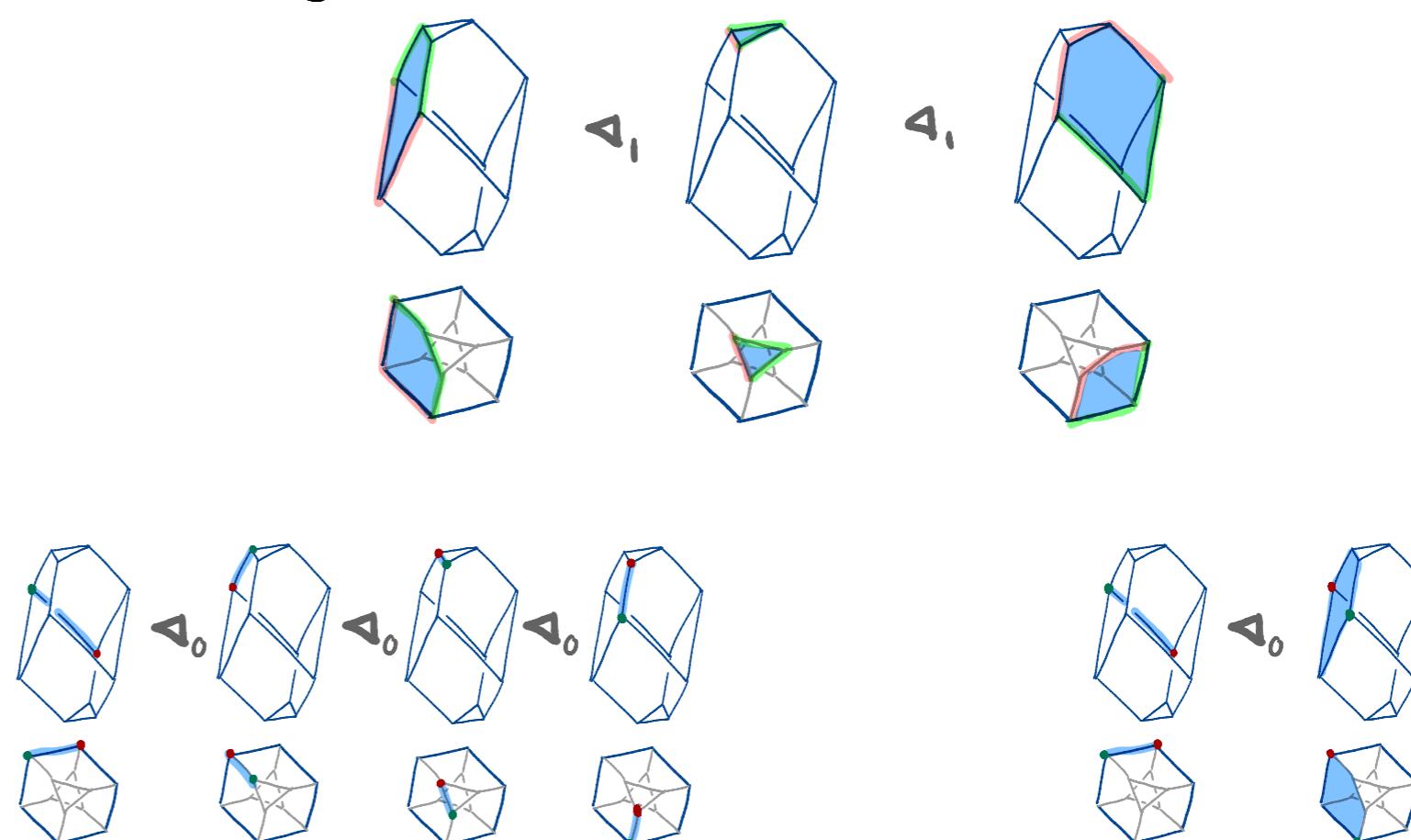
k -target: $t_k(F) = k$ -faces in upper envelope of $\pi_{k+1}(F)$



Strongly related to flag oriented matroids

CELLULAR STRINGS & LOOPS

Cellular k -string: $F_1 \triangleleft_k F_2 \triangleleft_k \dots \triangleleft_k F_m$ with $t_k(F_i) \cap s_k(F_{i+1}) \neq \emptyset$



Cellular loop: $F_i = F_j \quad i \neq j$

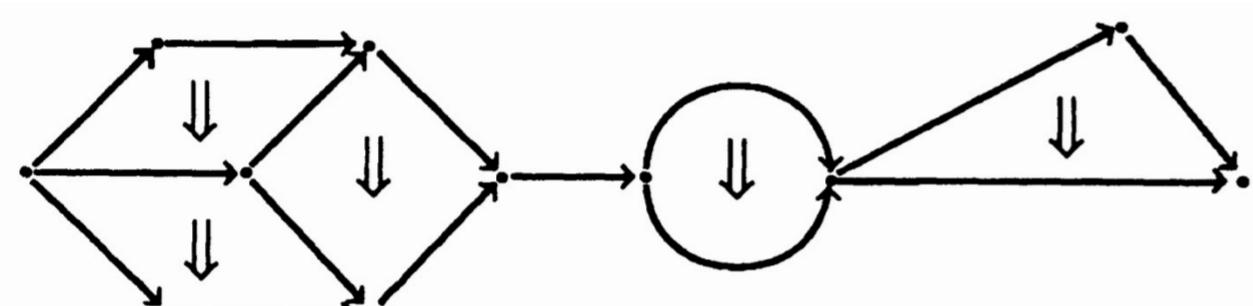
Cellular 0-strings from $s_0(P)$ to $t_0(P) = \text{Billera-Kapranov-Sturmfels' cellular strings}$

THE CONJECTURE

KAPRANOV & VOEVODSKY

COMBINATORIAL-GEOMETRIC ASPECTS OF POLYCATEGORY THEORY

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES
VOL. XXXII-1 (1991)



Conjecture:

Theorem 2.3. If $M \subset \mathbb{R}^n$ is a bounded n -dimensional polytope and $p = \{\mathbb{R}^n \xrightarrow{p_{n,n-1}} \mathbb{R}^{n-1} \xrightarrow{\dots} \mathbb{R}^2 \xrightarrow{p_{2,1}} \mathbb{R}\}$ is an admissible system of projections, then $A(M,p)$ is a composable pasting scheme.

↳ special type of d -dimensional category

We show that a framed polytope induces a composable pasting diagram \Leftrightarrow it has no cellular loops.

CONJECTURE (revisited): Every framed polytope is loop free.

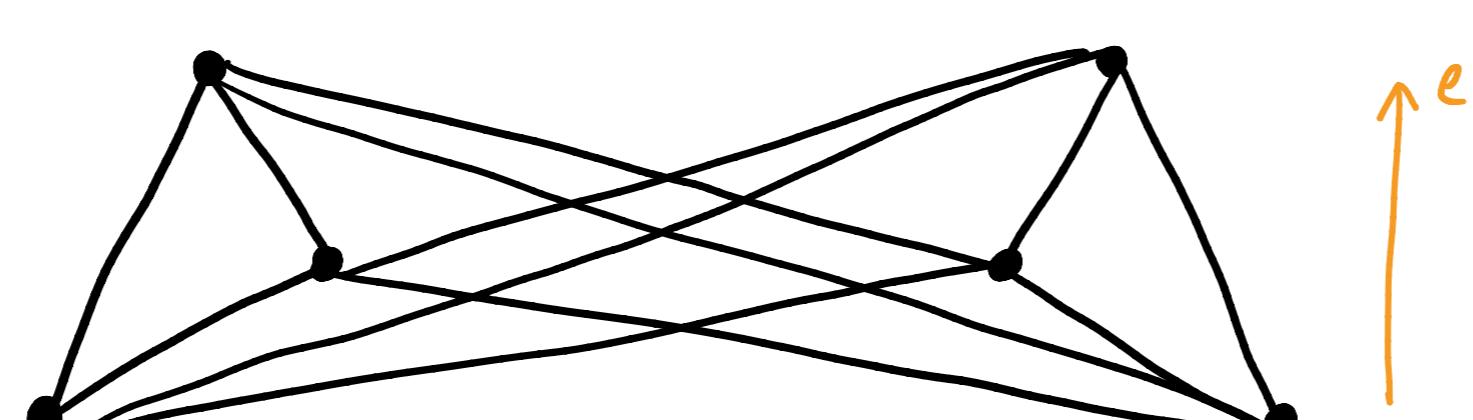
THE COUNTEREXAMPLES

A FIRST COUNTEREXAMPLE

The 5-simplex

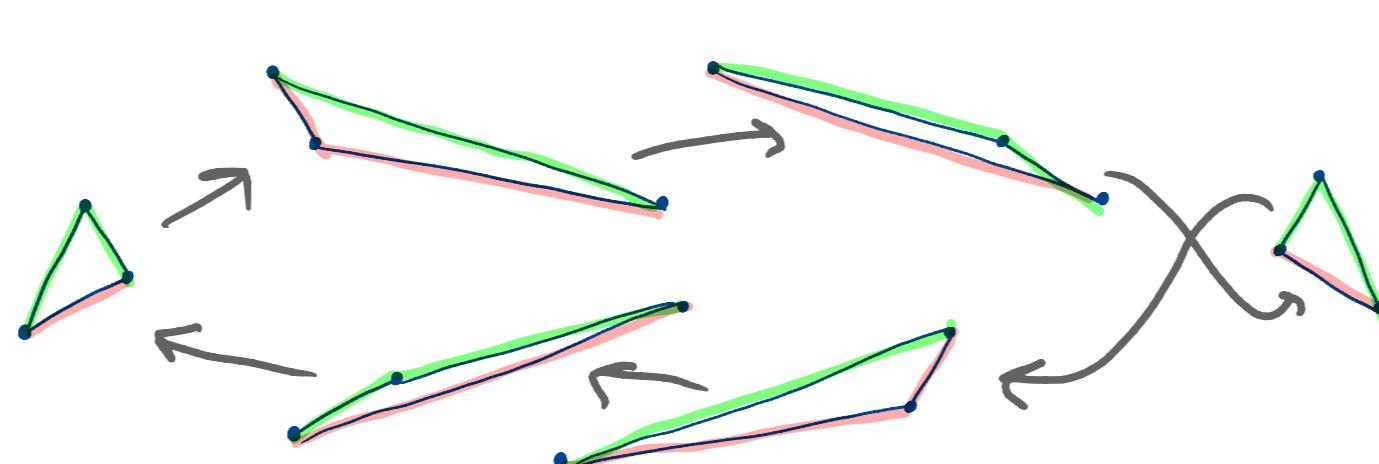
$$\text{conv} \left(\begin{pmatrix} -3 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

with the canonical frame has a 1-loop.



A STRONG COUNTEREXAMPLE

There is a 4-polytope with 144 vertices for which all admissible frames induce a loop.



RANDOM COUNTEREXAMPLES

A random Gaussian d -simplex with the canonical frame has a loop asymptotically almost surely when $d \rightarrow \infty$.