Projective dimension of weakly chordal graphic arrangements

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Hyperplane Arrangements

Definitions

Let $\mathbb{K}$ be a field and let V be a $\mathbb{K}$-vector space of dimension ℓ. A hyperplane H in V is a subspace of dimension $\ell - 1$. A hyperplane arrangement is a finite set of hyperplanes in V.

Let V^* be the dual space of V and $S = S(V^*)$ be the symmetric algebra of V^*. Identify S with $\mathbb{K}[x_1, .., x_\ell]$.

Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to a constant. The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

is called a defining polynomial of $\mathcal{A}$.

Let $V = \mathbb{K}^\ell$. Given a graph $G = (\mathcal{V}, E)$, define an arrangement $\mathcal{A}(G)$ by

$$\mathcal{A}(G) = \{\ker(x_i - x_j) | \{i,j\} \in E\}$$

called the graphic arrangement to G.

Deletion-Restriction

$L(\mathcal{A})$ is called the intersection lattice of the arrangement with partial order by reverse inclusion.

For $X \in L(\mathcal{A})$ define the subarrangement $\mathcal{A}_X$ of $\mathcal{A}$ by

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$$

as well as $(\mathcal{A}^X, X)$, an arrangement in X, by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \text{ and } X \cap H \neq \emptyset\}$$

called the restriction of $\mathcal{A}$ to X.

Let $\mathcal{A}$ be a non-empty arrangement and let $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \backslash \{H_0\}$ and let $\mathcal{A}'' = \mathcal{A}^{H_0}$.

The module of $\mathcal{A}$-Derivations (Saito '79)

A $\mathbb{K}$-linear map $\theta : S \to S$ is a derivation if for $f, g \in S$:

$$\theta(f \cdot g) = f \cdot \theta(g) + g \cdot \theta(f).$$

Let $\mathrm{Der}_{\mathbb{K}}(S)$ be the S-module of derivations of S.

Define an S-submodule of $\mathrm{Der}_{\mathbb{K}}(S)$, called the module of $\mathcal{A}$-derivations, by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

The arrangement $\mathcal{A}$ is called free if $D(\mathcal{A})$ is a free S-module.

Freeness of graphic arrangements [ER]

The following is known for graphic arrangements due to Stanley '72, Edelman, Reiner '94:

$\mathcal{A}(G)$ is free if and only if G is chordal.

Motivation: Terao's freeness conjecture ('83)

The freeness of an arrangement $\mathcal{A}$ defined over a fixed field $\mathbb{F}$ depends only on its intersection lattice $L(\mathcal{A})$, or equivalently, on its underlying matroid.

Projective dimension

Let M be a module. The projective dimension $\mathrm{pd}(M)$ is the minimum integer n (if it exists), such that there is a resolution of M by projective modules

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0.$$

The projective dimension of an arrangement is the projective dimension of its derivation module.

Lemma 1 [AKMM]

Assume that $\mathcal{A}_X$ is free for all $X \in L_2(\mathcal{A}^{H_0})$ and $pd(\mathcal{A}) \leq 1$. Then we also have $pd(\mathcal{A}') \leq 1$.

Our Contribution

Main result (Abe, Kühne, Mücksch, M. '23)

$\mathrm{pd}(\mathcal{A}(G)) = 1$ if and only if G is weakly chordal and non-chordal.

Proof outline

(1) Prove that if G is weakly chordal, then $\mathrm{pd}(\mathcal{A}(G)) \leq 1$.

(2) Prove that if G is a k-cycle, $k \geq 5$, then $\mathrm{pd}(\mathcal{A}(G)) > 1$.

(3) Prove that if G is a k-antihole, $k \geq 5$, then $\mathrm{pd}(\mathcal{A}(G)) > 1$.

This suffices because for any localization X, it holds that

$$\mathrm{pd}(\mathcal{A}_X) \leq \mathrm{pd}(\mathcal{A}).$$

Step (1)

Use Lemma 1 (hyperplane arrangement theory) iteratively to remove hyperplanes from the arrangement $\mathcal{A}(K_\ell)$, which is free.

Use Lemma 2 (graph theory) to prove that one can find a sequence of hyperplanes which iteratively meet the conditions of application of Lemma 1.

Step (2)

We show the following: *$pd(\mathcal{A}(C_k)) = k - 3$ for $k \geq 3$.*

In 2006, Kung and Schenck [KS] proved the following:

If G contains an induced cycle of length m, then $pd(\mathcal{A}(G)) \geq m - 3$.

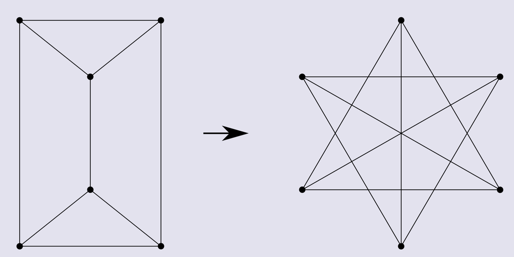

Figure 1: Triangular prism [KS] is the 6-antihole.

Step (3)

We prove: *For all $\ell \geq 6$ it holds that*

$$pd(\mathcal{A}(C_\ell^C)) = 2.$$

We found explicit generators for the module of derivations:

$$D(\mathcal{A}(C_\ell^C)) = \langle \theta_0, \ldots, \theta_{\ell-3}, \varphi_1, \ldots, \varphi_\ell \rangle_S.$$

where

$$\theta_i := \sum_{j=1}^{\ell} x_j^i \partial_{x_j} \ (i \geq 0) \text{ and } \varphi_i := \prod_{j \in [\ell] \backslash \{i-1, i, i+1\}} (x_i - x_j) \partial_{x_i}$$

which have relations, which in turn have a relation.

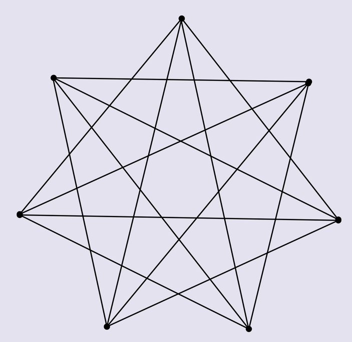

Figure 2: The 7-antihole

Outlook: other projective dimensions

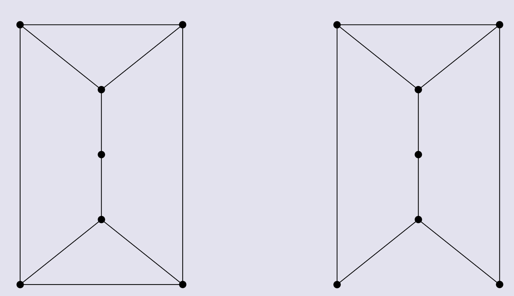

Figure 3: Two graphs on 7 vertices

The graphs in the pictures and their complement have longest chordless cycles of length 5, yet projective dimension 3.

Graph Theory

Definitions

A simple graph G on a set $\mathcal{V}$ is a tuple $(\mathcal{V}, E)$ with $E \subseteq \binom{\mathcal{V}}{2}$ the set of (undirected) edges connecting the vertices in $\mathcal{V}$. A graph $G' = (\mathcal{V}', E')$ with $\mathcal{V}' \subseteq \mathcal{V}, E' \subseteq E$ is called a subgraph of G. If $E' = \binom{\mathcal{V}'}{2} \cap E$, the graph G' is an induced subgraph of G.

A graph is chordal if it has no induced cycles of length > 3.

The graph $G^C = \left(\mathcal{V}, \binom{\mathcal{V}}{2} \backslash E\right)$ is called the complement graph of G.

Deletion-Contraction

Let $G = (\mathcal{V}, E)$ be a graph and $e = \{i, j\} \in E$.

(1) The graph $G' = (\mathcal{V}, E \backslash \{e\})$ is obtained from G through deletion of e.

(2) The graph $G'' = (\mathcal{V}'', E'')$ with $\mathcal{V}''$ the vertex set obtained by identifying i and j and $E'' = \{\{\bar{p}, \bar{q}\} \mid \{p, q\} \in E'\}$ is obtained by contraction of G with respect to e.

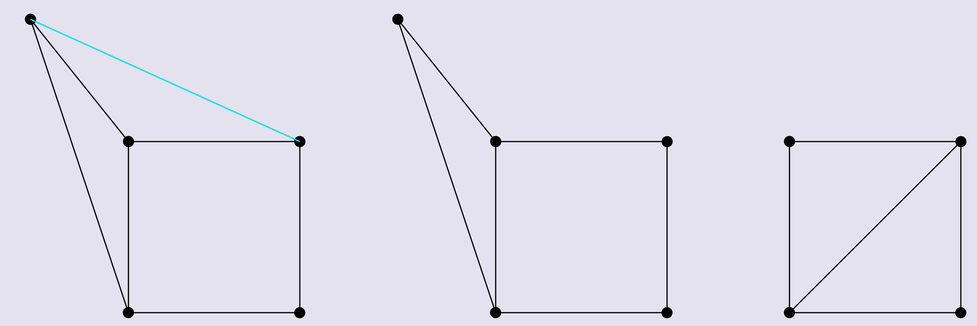

Figure 4:Example of deletion-contraction

Weakly chordal graphs [H]

The class of weakly chordal graphs was first introduced by Hayward in '83:

A graph is called weakly chordal if it contains no induced k-cycle with $k > 4$ and no complement of such a cycle as an induced subgraph.

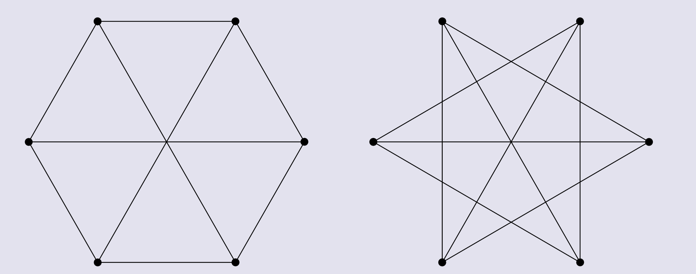

Figure 5: Weakly chordal graph (left) and the 6-antihole (right)

Lemma 2 [AKMM]

For a weakly chordal graph $G = (\mathcal{V}, E)$, there exists a sequence of edges $e_1, .., e_k \notin E$, such that

1. *$G_i = (\mathcal{V}, E \cup \{e_1, \ldots, e_i\})$ is weakly chordal for $i = 1, \ldots, k - 1$,*
2. *the edge e_i is not part of an induced cycle C_4 in G_i for $i = 1, \ldots, k$ and*
3. *G_k is chordal.*

References

[AKMM] T. Abe, L.Kühne, P. Mücksch and L. Mühlherr. *Projective dimension of weakly chordal graphic arrangements*, 2023. arXiv:2307.06021

[D] R. Diestel. *Graph theory*. Fifth. Vol. 173. Graduate Texts in Mathematics.

[ER] P. H. Edelman and V. Reiner. *Free hyperplane arrangements between An-1 and Bn*. In: Math. Z. 215.3 (1994), pp. 347-365 Springer, Berlin, 2017, pp. xviii+428.

[H] R. Hayward. *Weakly triangulated graphs*. In: J. Combin. Theory Ser. B 39.3 (1985), pp. 200–208.

[KS] J. P. S. Kung and H. Schenck. *Derivation modules of orthogonal duals of hyperplane arrangements*. J. Algebraic Combin.24(2006), no.3, 253–262.

[OT] P. Orlik and H. Terao. *Arrangements of hyperplanes* Vol. 300 Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 1992, pp. xviii+325

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