Rational Catalan Numbers for Complex Reflection Groups

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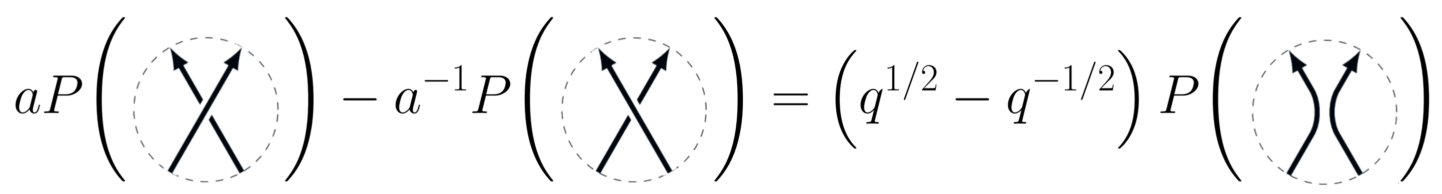
Knot Invariants

A fundamental problem in knot theory is determining whether two knot diagrams represent the same knot. One way to approach this problem is by finding *invariants* of a knot. For example, the number of *tricolorings*:

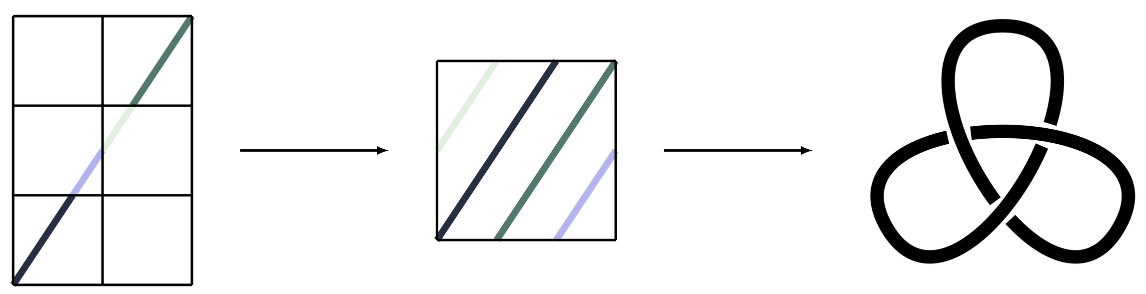
Colorings of a knot diagram with each intersection having either all three colors or only one color.

HOMFLYPT Polynomial — a sensitive knot invariant

The polynomial is defined by skein relations:



For n and p coprime, the *torus knot* $T_{n,p}$ winds n times around one S^1 factor and p times around the other.



For example, the trefoil is the $(2,3)$ torus knot and has HOMFLYPT polynomial

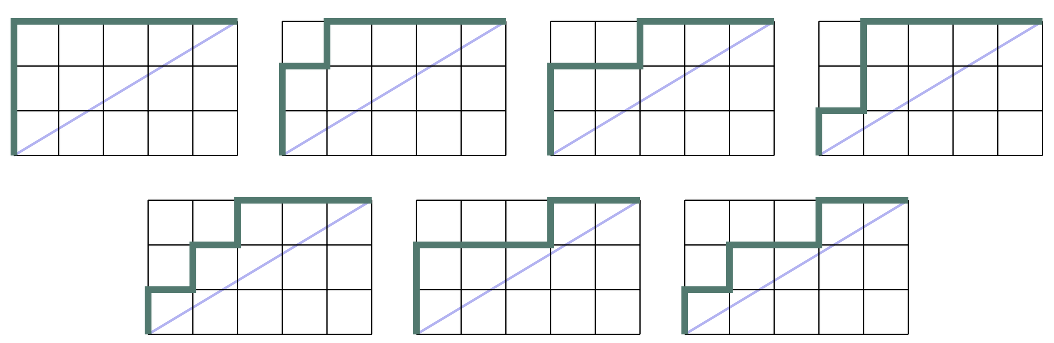
$$q^{-1}(q^2+1)a^{-2} - a^{-4}$$

The following is a result of Jones [3]. See also Gorsky [2].

$$[\text{top coefficient of } a]\ P(T_{n,p}) = \frac{1}{[n+p]_q}\begin{bmatrix} n+p \\ p \end{bmatrix}_q$$

Catalan Numbers — what do the torus knots know?

The *rational Catalan numbers* $\frac{1}{n+p}\binom{n+p}{p}$ count lattice paths in an $n \times p$ rectangle that stay above the diagonal.



Why do the torus knots "know" about Catalan numbers?

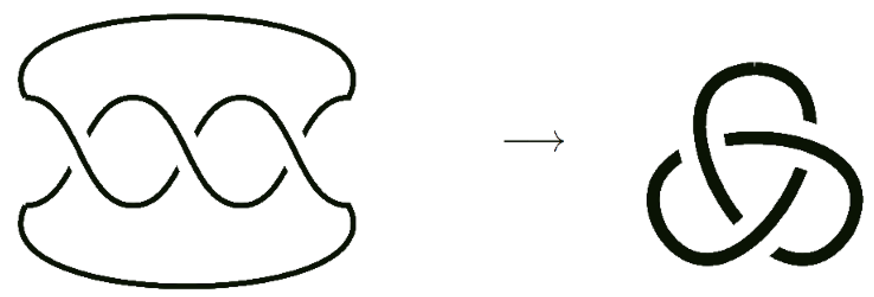
Braid Groups — knots from algebra

The *braid group* associated to the symmetric group has the following presentation:

$B_n = \langle \mathbf{s}_1, \ldots, \mathbf{s}_{n-1} : \mathbf{s}_i\mathbf{s}_j = \mathbf{s}_j\mathbf{s}_i$ for $|i-j| > 1$ and $\mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_i = \mathbf{s}_{i+1}\mathbf{s}_i\mathbf{s}_{i+1}\rangle$

This group can be pictured using *braids*, where the group operation is concatenation of diagrams.

Knots can be obtained by closing a braid. For example, the torus knot $T_{n,p}$ is the closure $\widehat{\beta}$ of the braid $\beta = (\mathbf{s}_1 \cdots \mathbf{s}_{n-1})^p$.



Hecke Algebra - HOMFLYPT from traces

$$\mathcal{H}_q = \mathbb{Q}(q^{\pm 1/2})[B_n]/\langle \mathbf{s}_i^2 = (q-1)\mathbf{s}_i + q\rangle$$

The Hecke algebra has a basis T_w indexed by elements of the symmetric group.

There is a trace function on the Hecke algebra which is able to detect the top coefficient of the HOMFLYPT polynomial:

$$\tau(T_\beta) := [T_1]T_\beta^{-1} = [\text{top coefficient of } a]P(\widehat{\beta})$$

So a hard geometric problem has been turned into a purely algebraic computation!

Braid Varieties — what is HOMFLYPT counting?

What is being counted by the trace? The number of points in a braid variety over a finite field [1]:

$$R_\beta(\mathbb{F}_q) := \left\{ B_+ \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} B_m \xleftarrow{w_0} B_- : B_i \in G/B_+ \right\}$$

This space can be decomposed into pieces indexed by special subwords. Taking "$q = 1$" and $\beta = (\mathbf{s}_1 \cdots \mathbf{s}_{n-1})^p$ shows that these subwords are counted by the Catalan numbers!

Complex Reflection Groups — HOMFLYPT without geometry

A *complex reflection group* is a subgroup of $GL(V)$ generated by reflections (a finite-order invertible linear transformation fixing a single hyperplane).

For example, the cyclic group of order m can be realized as a complex reflection group acting on 1-dimensional complex space, generated by multiplication by a primitive m-th root of unity.

Complex reflection groups have associated braid groups and Hecke algebras, as well as a canonical trace function. Moreover, there is an analogue of torus knots in the braid group called *Coxeter elements* and an analogue of Catalan numbers using natural invariants of the group.

So the algebraic trace computation still can be performed, even though the geometry of braid varieties no longer exists!

That is, one can compute the number of points in a nonexistent braid variety [4]:

$$\tau_q(T_{\mathbf{c}}^{-p}) = q^{-np}(1-q)^n \prod_{i=1}^{n} \frac{[p + (pe_i \bmod h)]_q}{[d_i]_q}$$

This algebraic computation still produces rational Catalan numbers!

A similar trace computation holds for complex reflection groups and is related to parking combinatorics in the real case:

$$\sum_{b \in \mathcal{B}} \tau_q(b^\vee T_{\mathbf{c}}^p b) = (q-1)^n [p]_q^n$$

An open problem is to find objects that are counted by this trace in the complex case.

References

[1] Pavel Galashin, Thomas Lam, Minh-Tâm Quang Trinh, and Nathan Williams. Rational noncrossing coxeter-catalan combinatorics. *arXiv preprint arXiv:2208.00121*, 2022.

[2] Eugene Gorsky. q, t-catalan numbers and knot homology. *arXiv preprint arXiv:1003.0916*, 2010.

[3] Vaughan FR Jones. Hecke algebra representations of braid groups and link polynomials. In *New Developments in the Theory of Knots*, pages 20–73. World Scientific, 1987.

[4] Weston Miller. Rational catalan numbers for complex reflection groups. *arXiv preprint arXiv:2310.12354*, 2023.