Vertex models for the product of a Schur and Demazure polynomial

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Abstract

The product of a Schur polynomial and Demazure atom or character expands positively in Demazure atoms or characters, respectively. The structure coefficients in these expansions have known combinatorial rules in terms of skyline tableaux [3]. We develop alternative rules using the theory of integrable vertex models, inspired by a technique introduced by Zinn-Justin [4]. We apply this method to coloured vertex models for atoms and characters obtained from Borodin and Wheeler's [1] models for non-symmetric Macdonald polynomials. The structure cofficients are then obtained as partition functions of vertex models that are compatible with both Schur (uncoloured) and Demazure (coloured) vertex models.

Demazure atoms and characters

- Demazure atoms (standard bases) refine characters, quasi-symmetric Schur polynomials and Schur polynomials
- Demazure characters (key polynomials) are characters of Demazure modules
- Both families are bases of $\mathbb{Z}[x_1, \ldots, x_n]$

Notation

- λ – integer partition: e.g. $(5, 5, 4, 2, 1, 0, 0)$
- α – weak composition: e.g. $(0, 1, 4, 0, 5, 5, 2)$
- Throughout, λ and α have length n and $x = (x_1, \ldots, x_n)$
- $s_\lambda(x)$ – Schur polynomial
- $\mathcal{A}_\alpha(x)$ – Demazure atom
- $\mathcal{K}_\alpha(x)$ – Demazure character

Structure coefficients

The products of a Schur polynomial with a Demazure polynomial have positive expansions:

$$s_\lambda(x)\mathcal{A}_\alpha(x) = \sum_\beta c^\beta_{\lambda,\alpha}\mathcal{A}_\beta(x)$$

$$s_\lambda(x)\mathcal{K}_\alpha(x) = \sum_\beta d^\beta_{\lambda,\alpha}\mathcal{K}_\beta(x),$$

where the ***structure coefficients*** $c^\beta_{\lambda,\alpha}$ and $d^\beta_{\lambda,\alpha}$ are non-negative integers. Haglund, Luoto, Mason, and van Willigenburg [3] found a combinatorial rule to compute these coefficients in terms of "skyline tableaux."

References

[1] A. Borodin and M. Wheeler. Nonsymmetric Macdonald polynomails via integrable vertex models. *Transactions of the American Mathematical Society*, 375:8353–8397, 2022.

[2] B. Brubaker, V. Buciumas, D. Bump, and H. P. A. Gustafsson. Colored five-vertex models and Demazure atoms. *Journal of Combinatorial Theory*, 178:159–195, 2021.

[3] J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg. Refinements of the Littlewood–Richardson rule. *Transactions of the American Mathematical Society*, 363(3):1665–1686, 2011.

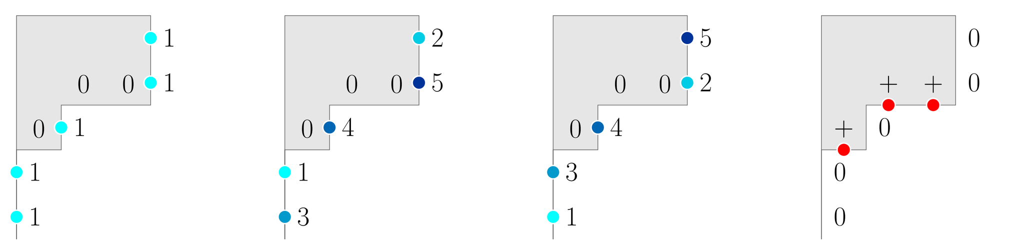
[4] P. Zinn-Justin. Littlewood–Richardson coefficients and integrable tilings. *The Electronic Journal of Combinatorics*, 16(1), 2009.

Vertex models for polynomials

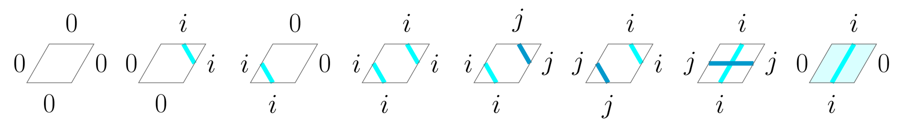
❶ Re-encode λ and α with strings

$$\lambda = (3, 3, 1, 0, 0)$$
$$\alpha = (0, 3, 0, 1, 3)$$

$\lambda^- = 11010011$ $\quad \alpha_{\mathcal{A}} = 31040052$ $\quad \alpha_{\mathcal{K}} = 13040025$ $\quad \lambda^+ = 00{+}0{+}{+}00$

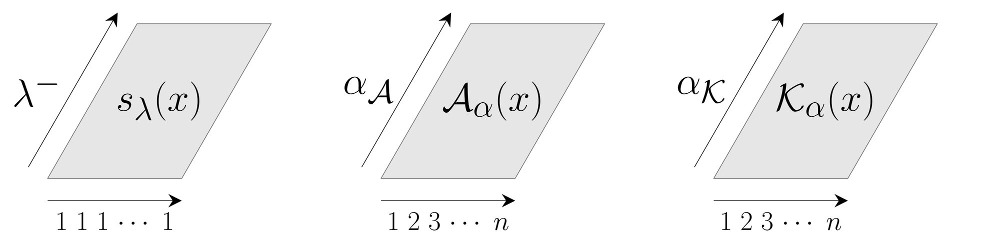


❷ Define tiles $(0 < i < j)$



(a) (b) weight: x_c

❸ Sum over weighted fillings of lattices to compute polynomials

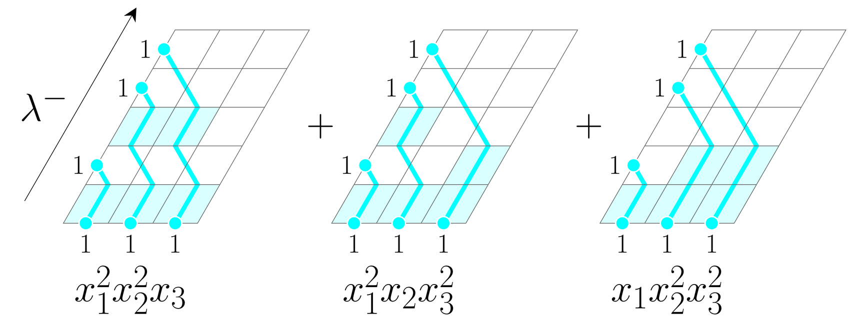


- $\mathcal{A}_\alpha(x)$ model excludes (a) tiles and $\mathcal{K}_\alpha(x)$ model excludes (b) tiles
- The integer c in x_c is the column number the tile occurs in
- The models for $s_\lambda(x)$ are the same as in Zinn-Justin's paper [4]
- The models for $\mathcal{A}_\alpha(x)$ and $\mathcal{K}_\alpha(x)$ are modifications of Borodin and Wheeler's non-symmetric Macdonald polynomial models (setting $q = t = 0$) [1]
- Brubaker, Buciumas, Bump and Gustafsson [2] also have a similar model for $\mathcal{A}_\alpha(x)$

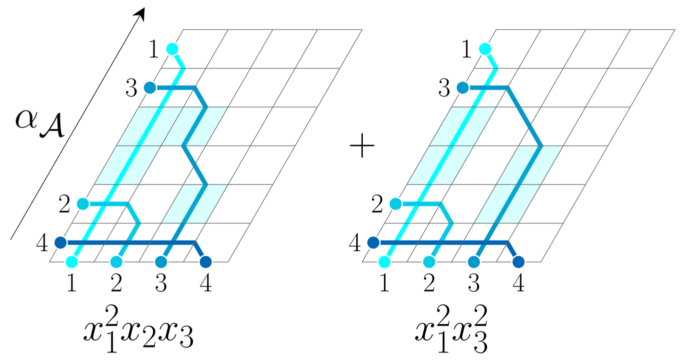
Model examples

$$\lambda = (2, 2, 1)$$
$$\alpha = (2, 0, 2, 0)$$

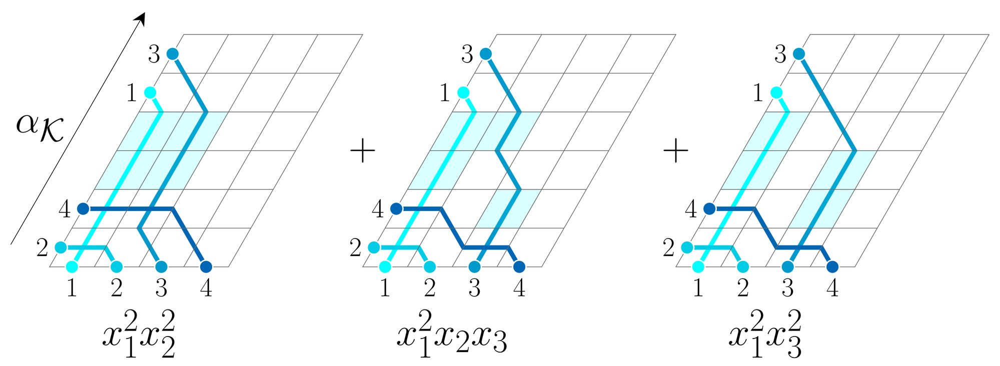
- $s_\lambda(x) = x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^2x_3^2$



- $\mathcal{A}_\alpha(x) = x_1^2x_2x_3 + x_1^2x_3^2$

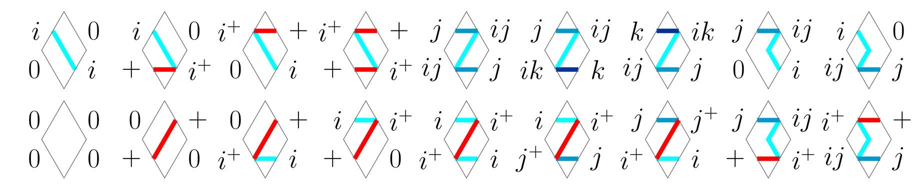


- $\mathcal{K}_\alpha(x) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2$

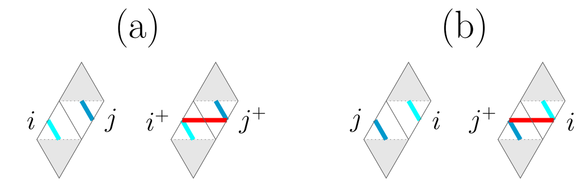


Theorem (M. 2024+)

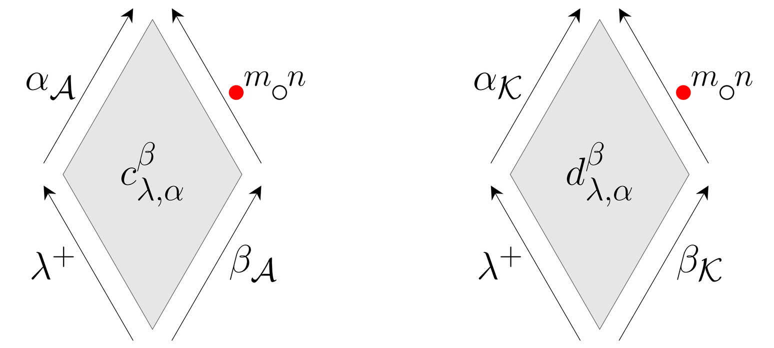
❶ Define diamond tiles $(0 < i < j < k)$



❷ Also prohibit certain adjacencies



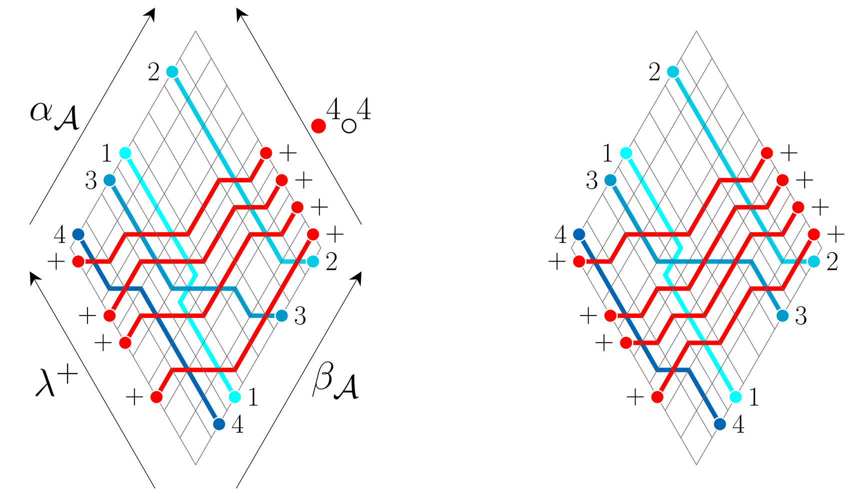
❸ The structure coefficients are computed by counting fillings (exclude adjacencies (a) in $c^\beta_{\lambda,\alpha}$ model and exclude (b) from $d^\beta_{\lambda,\alpha}$ model)



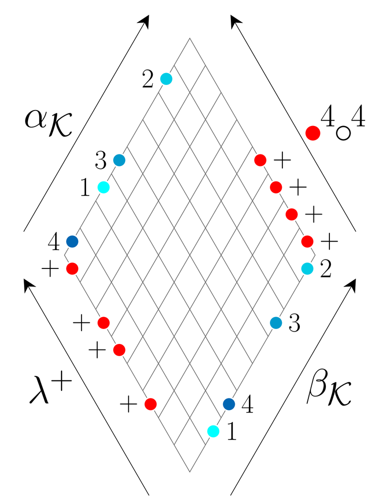
where $m \geq \max(\beta)$.

Diamond examples

- Let $\lambda = (3, 1, 0, 0)$, $\alpha = (1, 3, 1, 0)$ and $\beta = (1, 4, 3, 1)$. There are two fillings of the first model, so $c^\beta_{\lambda,\alpha} = 2$



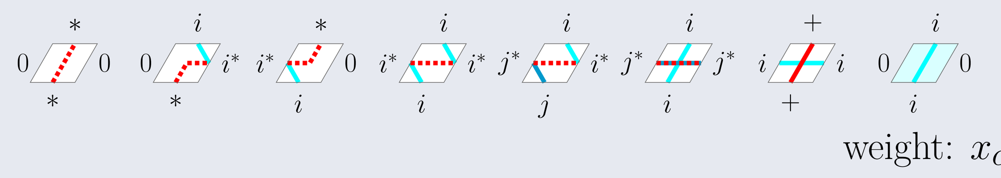
- There are no fillings of the second model, showing $d^\beta_{\lambda,\alpha} = 0$



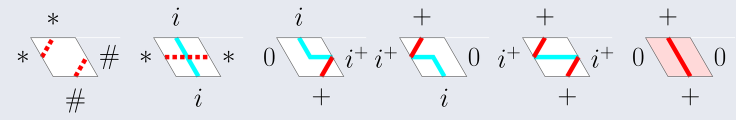
Proof of Theorem (Schur × atom)

New tiles

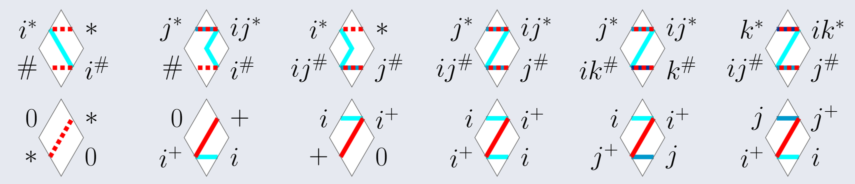
Invent new tiles (dotted lines may be red or white)



weight: x_c

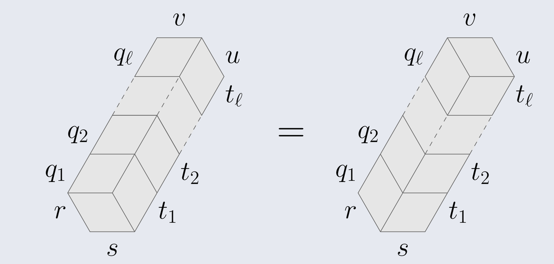


weight: $-x_c$

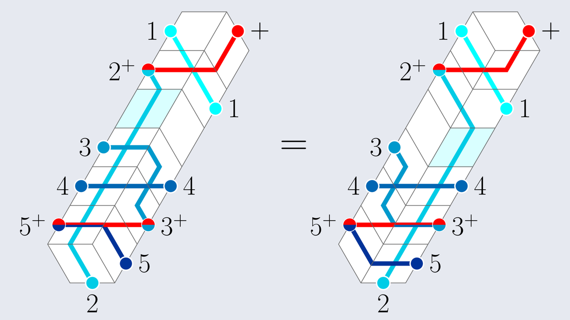


Column Lemma

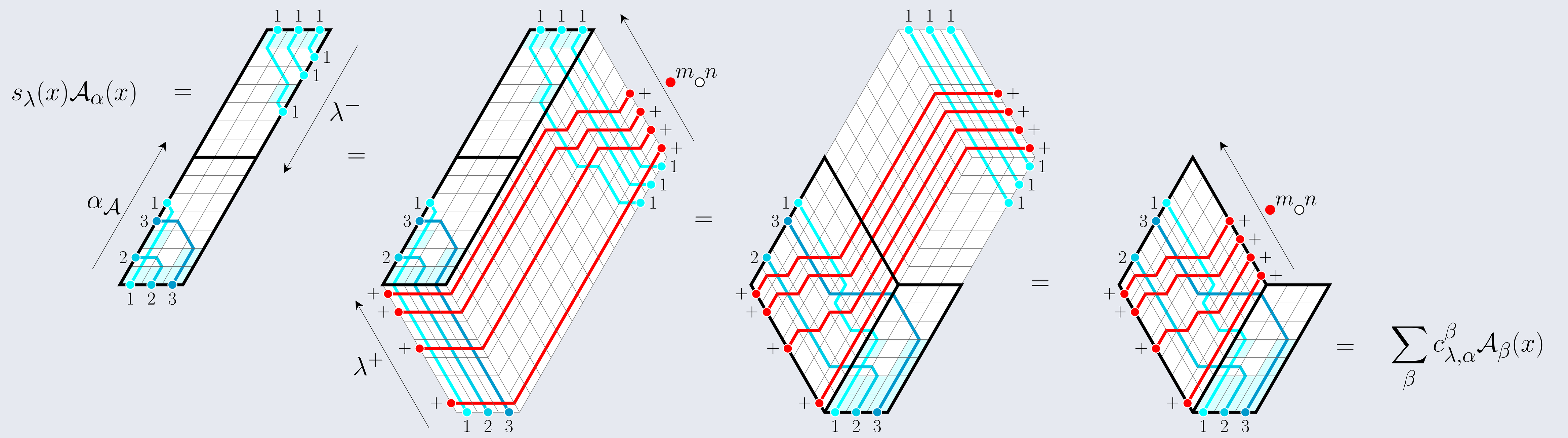
Can equate columns with same boundary if $r, u \in \{○, ●\}$



Example: both fillings have weight x_c



Proof: Repeatedly apply column lemma to get third equality



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