Fragmenting any Parallelepiped into a Signed Tiling

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Some Important Notation

$\Pi(N)$:= (half-open) fundamental paralellepiped of a matrix N

$\binom{[r+k]}{r}$:= size r subsets of $\{1, \dots, r+k\}$

$$\mathbb{1}_T(\mathbf{p}) := \begin{cases} 1 & \text{if } \mathbf{p} \in T, \\ 0 & \text{otherwise.} \end{cases}$$

S_σ, $\mathcal{T}_\sigma$, $\mathcal{T}^+$, and $\mathcal{T}^-$ are defined below.

Construction

1. Fix positive integers r and k as well as an $(r+k)\times(r+k)$ matrix M. We will be "fragmenting" the parallelepiped $\Pi(M)$.

2. Define M' to be M with the last k rows negated.

3. For each $\sigma \in \binom{[r+k]}{r}$, define the *fragment* S_σ to be the matrix obtained from M' by the following procedure:
   - For each column $i \in \sigma$, replace the bottom k entries with 0.
   - For each column $i \notin \sigma$, replace the top r entries with 0.

4. For each $\sigma \in \binom{[r+k]}{r}$, let
$$\mathcal{T}_\sigma := \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \Pi(S_\sigma) + M\mathbf{z}.$$
In other words, $\mathcal{T}_\sigma$ consists of all translations of the fundamental parallelepiped of $S_\sigma(M)$ by the lattice generated by M.

5. Define the sets
$$\mathcal{T}^+ := \bigsqcup_{\sigma\in\binom{[r+k]}{r},\ \det(S_\sigma)>0} \mathcal{T}_\sigma \quad\text{and}\quad \mathcal{T}^- := \bigsqcup_{\sigma\in\binom{[r+k]}{r},\ \det(S_\sigma)<0} \mathcal{T}_\sigma.$$
We call $\mathcal{T}^+$ the set of *positive tiles* and $\mathcal{T}^-$ the set of *negative tiles*.

Example 1

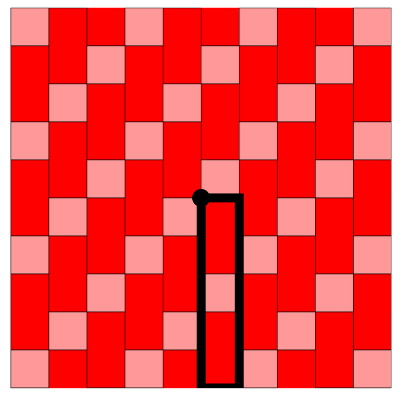
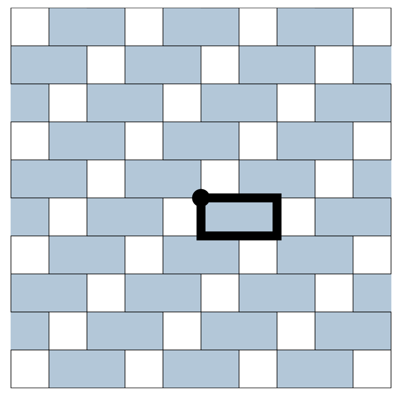
Let $r = k = 1$ and

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}. \text{ Then, } M' = \begin{bmatrix} 1 & 2 \\ -1 & -5 \end{bmatrix}.$$

There are 2 fragments:

$$S_{\{1\}} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} \text{ and } S_{\{2\}} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

Note that $\det(S_{\{1\}}) < 0 < \det(S_{\{2\}})$. Thus, $\mathcal{T}^+ = \mathcal{T}_{\{2\}}$ and $\mathcal{T}^- = \mathcal{T}_{\{1\}}$.

On the left are the tiles in $\mathcal{T}_{\{1\}}$ with $\Pi(S_{\{1\}})$ in bold. Each point is covered 1 or 2 times. On the right are the tiles in $\mathcal{T}_{\{2\}}$ with $\Pi(S_{\{2\}})$ in bold. Each point is covered 0 or 1 times.

In the language of our main theorem, for each point $\mathbf{p} \in \mathbb{R}^2$, we have

$$f(\mathbf{p}) = 1 - 2 = -1,$$

or

$$f(\mathbf{p}) = 0 - 1 = -1.$$

Furthermore,

$$(-1)^k \operatorname{sgn}(\det(M)) = -1.$$

Example 2

Let $r = k = 2$ and

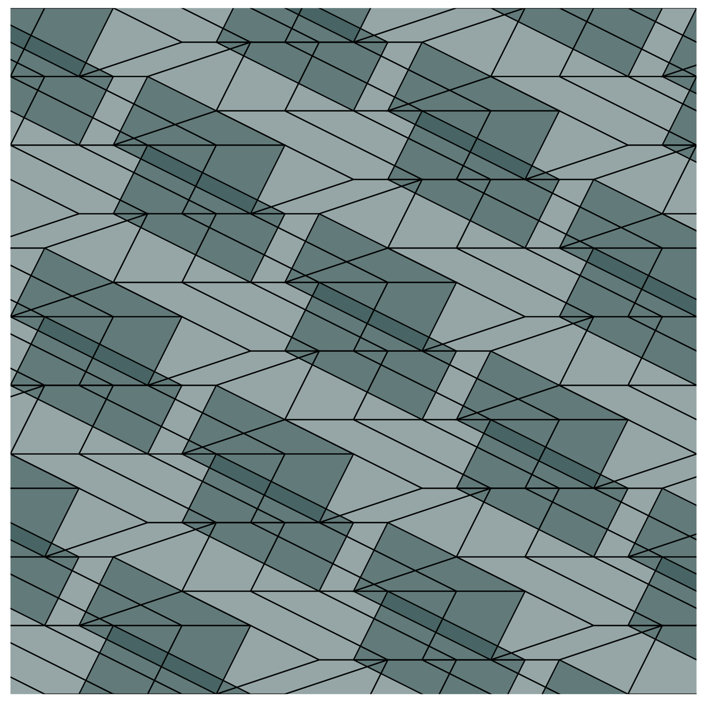
$$M = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \end{bmatrix}. \quad M' = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 1 & 0 & 2 & 2 \\ -2 & 0 & 1 & -1 \\ 0 & -1 & 2 & -3 \end{bmatrix}$$

There are 6 fragments. Here are 2 examples:

$$S_{\{1,2\}} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -3 \end{bmatrix} \quad S_{\{2,3\}} = \begin{bmatrix} 0 & 2 & -4 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

One can show that $\det(S_\sigma) > 0$ when $\sigma \neq \{3,4\}$ and $\det(S_{\{3,4\}}) < 0$. This means that:

$$\mathcal{T}^+ = \mathcal{T}_{\{1,2\}} \sqcup \mathcal{T}_{\{1,3\}} \sqcup \mathcal{T}_{\{1,4\}} \sqcup \mathcal{T}_{\{2,3\}} \sqcup \mathcal{T}_{\{2,4\}}$$
and $\mathcal{T}^- = \mathcal{T}_{\{3,4\}}$.



Above is a 2-dimensional slice of $\mathcal{T}^+$. Each point in $\mathbb{R}^4$ is covered by 1, 2, or 3 positive tiles indicated by the shading. Can you deduce from our main theorem what a 2-dimensional slice of $\mathcal{T}_{\{3,4\}}$ looks like?

Our Main Theorem

For any matrix M, the function $f(\mathbf{p}) : \mathbb{R}^{r+k} \to \mathbb{Z}$, defined by

$$f(\mathbf{p}) := \left(\sum_{T\in\mathcal{T}^+(M)} \mathbb{1}_T(\mathbf{p})\right) - \left(\sum_{T\in\mathcal{T}^-(M)} \mathbb{1}_T(\mathbf{p})\right),$$

is constant with value $(-1)^k \operatorname{sgn}(\det(M))$.

Sketch of Proof Ideas

First, we use the Multi-Row Laplace Expansion Formula to find that $(-1)^k \operatorname{sgn}(\det(M))$ is the average value of f. Thus, we "just" need to show f is constant.

To show that f is constant, we imagine a particle moving in a generic direction. We must prove that whenever the particle *enters* a *positive* tile or *exits* a *negative* tile, it also *exits* a *positive* tile or *enters* a *negative* tile.

To prove this, we find a natural way to group collections of *facets* of tiles such that the property from the previous paragraph holds for each group. This proof uses a variety of linear algebra techniques (including two versions of *Cramer's Rule*), but the main challenge was developing useful notation to keep track of everything.

Corollary [2]

If M is invertible, and $\det(S_\sigma(M))$ is nonnegative for all $\sigma \in \binom{[r+k]}{r}$ (or nonpositive for all $\sigma \in \binom{[r+k]}{r}$), then

$$\mathbb{R}^{r+k} = \bigsqcup_{\sigma\in\binom{[r+k]}{r}} \mathcal{T}_\sigma.$$

References

[1] Joseph Doolittle and Alex McDonough. Fragmenting any parallelepiped into a signed tiling. *Discrete & Computational Geometry*, pages 1–34, 2024.

[2] Alex McDonough. *Higher-Dimensional Sandpile Groups and Matrix-Tree Multijections*. PhD thesis, Brown University, 2021.