Kromatic quasisymmetric functions

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1. Chromatic/kromatic symmetric/quasisymmetric functions

- All graphs $G = (V, E)$ have finite vertices and simple edges, and are not distinguished up to $\cong$. A *proper coloring* of a graph $G = (V, E)$ is a map $\kappa : V \to \mathbb{Z}_{>0}$ with $\kappa(u) \neq \kappa(v)\ \forall \{u, v\} \in E$.

Definition (Stanley, 1995)

The *chromatic symmetric function* of a graph $G = (V, E)$ is the sum over all proper colorings

$$X_G = \sum_\kappa x^\kappa = \sum_\kappa \prod_{v\in V} x_{\kappa(v)} \in \mathsf{Sym} = (\text{ bounded degree symmetric functions over } \mathbb{Z}\ ).$$

- We say that $G = (V, E)$ is an *ordered graph* if there is a total order $<$ on the vertex set V. In this case let $\mathsf{asc}_G(\kappa)$ be the number of edges $\{u, v\} \in E$ with $u < v$ and $\kappa(u) < \kappa(v)$.

Definition (Shareshian–Wachs, 2016)

The *chromatic quasisymmetric function* of an ordered graph G is the sum over all proper colorings

$$X_G(q) = \sum_\kappa q^{\mathsf{asc}_G(\kappa)} x^\kappa \in \mathsf{QSym}[q] = (\text{ quasisymmetric functions over } \mathbb{Z}[q]\ ).$$

- A *(proper) set-valued coloring* of G is a map $\kappa : V \to \{\text{finite nonempty subsets of integers} > 0\}$ with $\kappa(u) \cap \kappa(v) = \varnothing$ for all edges $\{u, v\} \in E$.

Definition (Crew–Pechenik–Spirkl, 2023)

The *kromatic symmetric function* of a graph $G = (V, E)$ is the sum over all set-valued colorings

$$\overline{X}_G = \sum_\kappa x^\kappa = \sum_\kappa \prod_{v\in V} \prod_{i\in\kappa(v)} x_i \in \mathsf{mSym} = (\text{ arbitrary symmetric functions over } \mathbb{Z}\ ).$$

- If $G = K_n$ is the *complete graph* on $\{1, 2, \dots, n\}$ and $e_n = \sum_{1\le i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ then

$$X_G = n!\cdot e_n \quad\text{and}\quad X_G(q) = [n]_q!\cdot e_n \quad\text{and}\quad \overline{X}_G = n!\cdot \sum_{r\ge n} \left\{ {r \atop n} \right\} e_r.$$

- *Stanley–Stembridge conjecture*: if P is a $(3 + 1)$-free poset and $G = \mathsf{inc}(P)$ then X_G is e-positive.

2. Combinatorial Hopf algebras

- Let $\mathbb{K}$ be an integral domain like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}[q]$, or $\mathbb{Q}(q)$.
- An *algebra* is a $\mathbb{K}$-module A with linear product $\nabla : A \otimes A \to A$ and unit $\iota : \mathbb{K} \to A$ maps.

  A *coalgebra* is a $\mathbb{K}$-module A with linear coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$ maps.

  The (co)product and (co)unit maps must satisfy several natural associativity axioms.
- A $\mathbb{K}$-module A that is both an algebra and a coalgebra is a *bialgebra* if the coproduct and counit maps are algebra morphisms, and a *Hopf algebra* if it further holds that the identity map

$$\mathrm{id} : A \to A$$

  has a two-sided inverse in the *convolution algebra*, which is the set of $\mathbb{K}$-linear maps $A \to A$ with

$$\text{unit element } \iota \circ \epsilon \qquad\text{and}\qquad \text{product } f * g = \nabla \circ (f \otimes g) \circ \Delta.$$

Definition (mildly generalizing Aguiar–Bergeron–Sottile, 2006)

A *combinatorial Hopf algebra* is a Hopf algebra H with an algebra morphism $\zeta : H \to \mathbb{K}[\![t]\!]$ such that

$$\zeta(\cdot)|_{t\mapsto 0} = (\text{ the counit of } H\ ).$$

A *morphism* $(H, \zeta) \to (H', \zeta')$ is a Hopf algebra morphism $\phi : H \to H'$ with $\zeta = \zeta' \circ \phi$.

- A power series $f \in \mathbb{K}[\![x_1, x_2, \dots]\!]$ is *quasisymmetric* if

$$[x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}]f = [x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}]f \quad \text{for all } 1 \le i_1 < i_2 < \cdots < i_k \text{ and } \alpha \in \mathbb{Z}^k.$$

- The algebra of quasisymmetric functions QSym over $\mathbb{K}$ has a combinatorial Hopf algebra structure for the algebra morphism $\zeta_{\mathsf{QSym}} : \mathsf{QSym} \to \mathbb{K}[\![t]\!]$ setting $x_1 \mapsto t$ and all other variables to zero.

  The coproduct of QSym corresponds informally to the "variable doubling operation"

$$\Delta : f(x_1, x_2, \dots) \mapsto f(x_1, x_2, \dots, y_1, y_2, \dots).$$

Theorem (mildly generalizing Aguiar–Bergeron–Sottile, 2006)

There is a unique morphism $\Phi : (H, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ for each combinatorial Hopf algebra (H, ζ).

3. Hopf algebraic constructions

- The free module Graphs of finite simple graphs (up to $\cong$) is a combinatorial Hopf algebra with

$$\nabla(G_1 \otimes G_2) = G_1 \sqcup G_2, \qquad \Delta(G) = \sum_{S\sqcup T=V(G)} G|_S \otimes G|_T, \qquad \zeta(G) = 0^{|E(G)|}t^{|V(G)|}.$$

  The coproduct sums over **disjoint** partitions of vertices, with $G|_S$ denoting the induced subgraph.

Proposition (Aguiar–Bergeron–Sottile, 2006)

The unique morphism $(\mathsf{Graphs}, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ sends each graph $G \mapsto X_G$.

- The free module $\mathsf{OGraphs}$ of ordered graphs (up to $\cong$) is likewise a combinatorial Hopf algebra with

$$\nabla(G_1 \otimes G_2) = G_1 \sqcup G_2, \quad \Delta(G) = \sum_{S\sqcup T=V(G)} q^{\mathsf{asc}_G(S,T)} G|_S \otimes G|_T, \quad \zeta(G) = 0^{|E(G)|}t^{|V(G)|}$$

  when $\mathsf{asc}_G(S, T) = |\{(s, t) \in S \times T : \{s, t\} \in E(G) \text{ and } s < t\}|$.

Proposition

The unique morphism $(\mathsf{OGraphs}, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ sends each ordered graph $G \mapsto X_G(q)$.

- There is a similar construction of $\overline{X}_G$ using combinatorial Hopf algebras.

Theorem

Suppose we redefine the coproduct of Graphs to be the sum over arbitrary (rather than disjoint) unions:

$$\Delta(G) = \sum_{S\cup T=V(G)} G|_S \otimes G|_T.$$

Then the unique morphism $(\mathsf{Graphs}, \zeta) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$ sends each graph $G \mapsto \overline{X}_G$.

Moreover, this modified Hopf algebra structure on Graphs is "isomorphic" to the original one.

- Technical caveats: for the isomorphism just mentioned to hold, we must replace Graphs with its *completion* $\mathfrak{m}\mathsf{Graphs}$ (whose elements are infinite linear combinations of graphs) and work with the analogues of Hopf algebras in the category of *linearly compact $\mathbb{K}$-modules*.

Corollary

$\overline{X}_G$ expands positively into the *multifundamental quasisymmetric functions* $\overline{L}_\alpha$ of Lam–Pylyavskyy.

4. Quasisymmetric versions of kromatic symmetric functions

- Assume G is an ordered graph. There are two simultaneous generalizations of $X_G(q)$ and $\overline{X}_G$. Neither comes from a combinatorial Hopf algebra but both are natural, in different ways.

Definition

For an ordered graph G let $\overline{L}_G(q) = \sum_\kappa q^{\mathsf{asc}_G(\max\circ\kappa)} x^\kappa \in \mathsf{QSym}[q]$, summing over set-valued colorings.

- Changing ascents-to-descents or max-to-min in this definition give up to 4 different power series, but the distinct constructions are all related by simple automorphisms of $\mathsf{QSym}[q]$.
- If $G = K_n$ is the complete graph on $\{1, 2, \dots, n\}$ then $\overline{L}_G(q) = [n]_q!\cdot \sum_{r\ge n} \left\{ {r \atop n} \right\} e_r$.

Theorem

The power series $\overline{L}_G(q)$ is always *multifundamental positive*, with a particularly simple mulifundamental expansion into $\overline{L}_\alpha$'s when $G = \mathsf{inc}(P)$ is the incomparability graph of a finite poset $P \subset \mathbb{Z}$.

Theorem

However, the power series $\overline{L}_G(q)$ is symmetric in its x_i-variables if and only if G is a *cluster graph*.

- The second QSym-version of $\overline{X}_G$ is related to functions $X_G(\mathbf{x}, q, \mu)$ studied by Hwang (2022).

Definition

For an ordered graph G consider the sum over all set-valued colorings

$$\overline{X}_G(q) = \sum_\kappa q^{\mathsf{asc}_G(\kappa)} x^\kappa = \sum_{\mu:V(G)\to\mathbb{Z}_{>0}} X_G(\mathbf{x}, q, \mu) \in \mathsf{QSym}[\![q]\!]$$

where $\mathsf{asc}_G(\kappa)$ counts all (u, v, i, j) with $\{u, v\} \in E(G)$, $i \in \kappa(u)$, $j \in \kappa(v)$, $u < v$, and $i < j$.

- In general $\overline{X}_G(q)$ is not multifundamental-positive, but it does have a different positivity property.
- A *natural unit interval order* is a poset P on a finite subset of $\mathbb{Z}_{>0}$ that is $(3 + 1)$- and $(2 + 2)$-free. Assume P is such a poset. When $q = 1$ the following recovers a theorem of Crew–Pechenik–Spirkl:

Theorem

If $G = \mathsf{inc}(P)$ is the incomparability graph of a natural unit interval order P then $\overline{X}_G$ has an explicit positive expansion into *symmetric Grothendieck functions* $\overline{s}_\lambda = \sum_{T\in\mathsf{SetSSYT}(\lambda)} (-1)^{|T|-|\lambda|} x^T$.

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