FROM POSET POLYTOPES AND PIPE DREAMS TO FLAG VARIETIES AND REPRESENTATIONS

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TYPE A

The poset polytopes

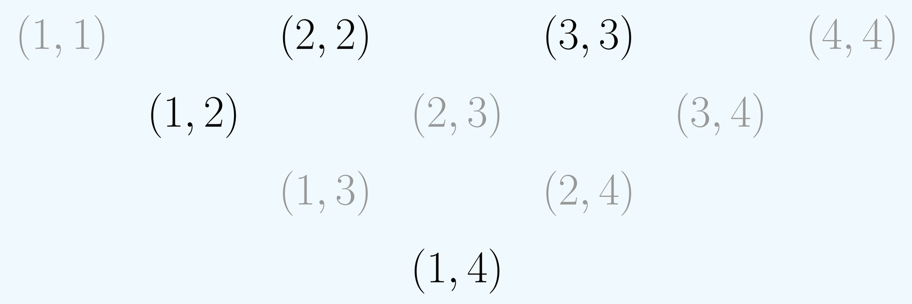
Gelfand–Tsetlin poset: $P = \{(i,j)\}_{1\le i\le j\le n}$ with order relation

$$(i,j) \preceq (i',j') \text{ if and only if } i \le i' \text{ and } j \le j'.$$

Fix $O \subset P$. For lower set $J \subset P$ we define

$$M_O(J) = (J \cap O) \cup \max_{\prec}(J).$$

Example: for $O = \{(1,2),(2,2),(3,3)\}$,
$J = \{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3),(1,4)\}$ we get



Marked chain-order polytopes (Fang, Fourier): for $\mathfrak{sl}_n(\mathbb{C})$ fundamental weight ω_k let $\mathcal{Q}_O(\omega_k)$ be the convex hull of all $\mathbf{1}_{M_O(J)} \in \mathbb{R}^P$ s.t. $k = k(J)$ (number of $(i,i) \in J$). For dominant $\lambda = (a_1,\dots,a_{n-1})$ set

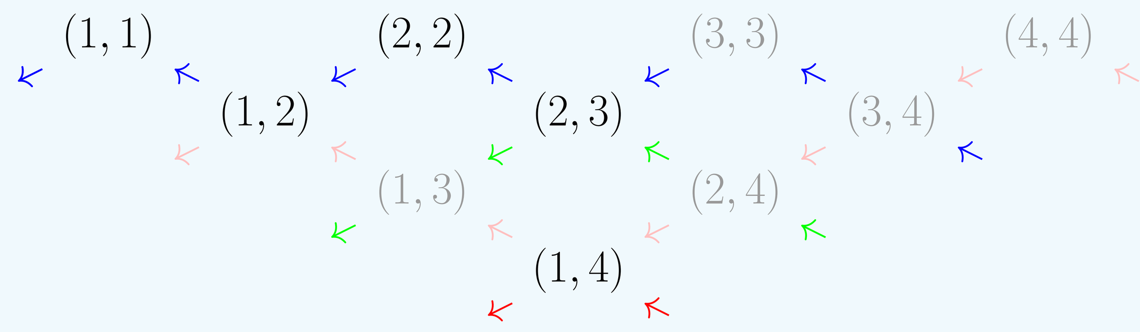
$$\mathcal{Q}_O(\lambda) = a_1\mathcal{Q}_O(\omega_1) + \dots + a_{n-1}\mathcal{Q}_O(\omega_{n-1}) \subset \mathbb{R}^P.$$

For $O = P$ this is the **Gelfand–Tsetlin polytope**, for $O = \varnothing$ it is the **FFLV polytope**, other cases interpolate between these two.

The key bijection

Pipe dreams: every $M \subset P$ defines permutation $w_M = \prod_{(i,j)\in M}(i\,j)$ in S_n (ordered first by i and then by j). In fact, if one draws a "pipe" entering (i,n) and then turning in zigzag fashion at elements of M and at the (j,j), then it will exit at $(1, w_M(i))$. These pipes form the *pipe dream* of M.

Example: $O = \{(1,1),(2,2),(1,2),(1,4)\}$,
$J = \{(1,1),(2,2),(1,2),(2,3),(1,3),(1,4)\}$,
$w_{M_O(J)} = (1\,1)(1\,2)(1\,4)(2\,2)(2\,3) = (4,3,1,2)$ is found from the pipe dream



Theorem. $J \mapsto \{w_{M_O(J)}(1),\dots,w_{M_O(J)}(k(J))\}$ is a bijection between lower sets in P and subsets of $[1,n]$.

Let us consider a version of this map but to ordered tuples and twisted by w_O:

$$\psi : J \mapsto (w_O^{-1}w_{M_O(J)}(1),\dots,w_O^{-1}w_{M_O(J)}(k(J))).$$

APPLICATIONS TO LIE THEORY

Toric degenerations

The **Plücker ideal** I in $S = \mathbb{C}[X_{i_1,\dots,i_k}]_{1\le i_1<\dots<i_k\le n,\,k\in[1,n-1]}$ gives a multiprojective realization of the (complete) flag variety F_n.

The **toric variety of** $\mathcal{Q}_O(\lambda)$ is realized by a toric ideal I_O in the ring $R = \mathbb{C}[X_J]_{k(J)\in[1,n-1]}$.

Theorem. The isomorphism $X_J \mapsto X_{\psi(J)}$ from R to S maps I_O to an initial ideal of I. This realizes the toric variety of $\mathcal{Q}_O(\lambda)$ as a flat degeneration of F_n.

For $O = P$ this is the **Gonciulea–Lakshmibai degeneration**.

PBW-monomial bases

Positive roots of A_{n-1} are labeled by pairs $1 \le i < j \le n$, let $f_{i,j} \in \mathfrak{sl}_n(\mathbb{C})$ be the corresponding negative root vector. A point $x \in \mathbb{Z}^P_{\ge 0}$ defines a PBW monomial $f^x = \prod f_{i,j}^{x_{i,j}}$ (ordered first by i and then by j).

Lemma. For every J and $i \in [1,k(J)]$ one has $\psi(J)_i \ge i$. Furthermore, one can define a unimodular transformation ξ of $\mathbb{Z}^P$ such that $\xi(\mathbf{1}_{M_O(J)}) = \mathbf{1}_{\{(i,\psi(J)_i),\,i\in[1,k(J)]\}}$ for every J.

Theorem. The vectors $f^x v_\lambda$ with $x \in \xi(\mathcal{Q}_O(\lambda)) \cap \mathbb{Z}^P$ form a basis in the irreducible representation V_λ with highest weight v_λ.

For $O = \varnothing$ this is the **FFLV basis**.

Standard monomial theories

Theorem. Products $X_{\psi(J_1)}\dots X_{\psi(J_m)}$ with $J_1 \supset \dots \supset J_m$ map to a basis in the Plücker algebra S/I.

Each such **standard monomial** defines a Young tableau with columns $\psi(J_i)$:

$\psi(J_1)_1$	$\psi(J_2)_1$	...	$\psi(J_m)_1$
...	...	...	...
...	$\psi(J_2)_{k(J_2)}$	...	
$\psi(J_1)_{k(J_1)}$			

For $O = P$ these are the **semistandard tableaux**, for $O = \varnothing$ one gets the **PBW-semistandard tableaux**.

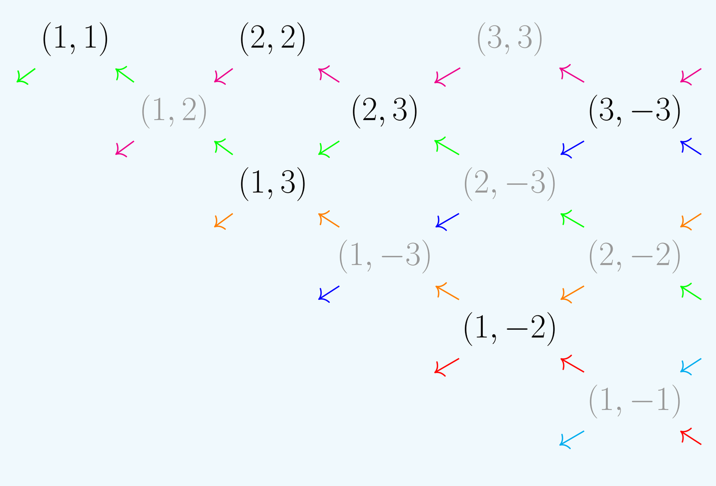
TYPE C

Combinatorics

For type C the poset P consists of (i,j) with $i \in [1,n]$ and $j \in [i,n] \cup [-n,-i]$ and we set $(i_1,j_1) \preceq (i_2,j_2)$ if and only if $i_1 \le i_2$ and j_1 precedes j_2 in the order $1,\dots,n,-n,\dots,-1$.

For $O \subset P$ and lower set J we define $M_O(J)$ and $w_{M_O(J)}$ similarly to type A, the values $w_{M_O(J)}(\pm i)$ are determined by the two pipes entering $(i,-i)$.

Example: $O = \{(1,1),(2,2),(2,3),(1,3)\}$, J generated by $(3,-3)$, $(1,-2)$,
$w_{M_O(J)}(1,2,3,-3,-2,-1) = (-2,1,-3,2,3,-1)$



Theorem. The map $J \mapsto \{w_O^{-1}w_{M_O(J)}(1),\dots,w_O^{-1}w_{M_O(J)}(k(J))\}$ is a bijection between lower sets in P and *admissible subsets* of $A \subset [1,n]$: those for which $|A \cap [-i,i]| \le i$ for all $i \in [1,n]$.

We also define the polytopes $\mathcal{Q}_O(\lambda)$ as in type A: for fundamental weights as convex hulls of the $\mathbf{1}_{M_O(J)}$ and for dominant weights as Minkowski sums. This again interpolates between the symplectic GT and FFLV polytopes.

Lie theory

- In type C the Plücker ideal lies in the polynomial ring in variables $X_{i_1,\dots,i_k}$ with $\{i_1,\dots,i_k\}$ an admissible subset. This lets us define **toric degenerations** of the symplectic flag variety and **standard monomial theories** for the corresponding Plücker algebra.
- An important feature of type C is the intermediate degeneration of the flag variety which is itself a **Schubert variety of type** A_{2n-1}. This Schubert variety subsequently degenerates into the toric varieties of all $\mathcal{Q}_O(\lambda)$.
- Furthermore, type C_n positive roots are enumerated by P. This lets us associate PBW monomials with points of $\mathbb{R}^P$ and obtain a **PBW-monomial basis** in V_λ given by a unimodular transform of $\mathcal{Q}_O(\lambda)$.
- Moreover, one can realize the $\mathcal{Q}_O(\lambda)$ as **Newton–Okounkov bodies** of the flag variety (this is also true in type A).
- Special cases for $O = P$ and $O = \varnothing$: Caldero's toric degeneration given by the type C Berenstein–Zelevinsky polytope, de Concini's symplectic SSYTs, the symplectic FFLV basis and Kaveh's Newton–Okounkov bodies.