# Ehrhart polynomials, Hecke series, and affine buildings

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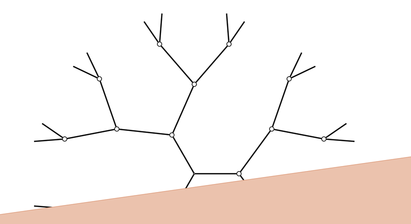
## Affine building for $\mathrm{SL}_n(\mathbb{Q}_p)$

$n-1$-dimensional simplicial complex on homothety classes of lattices $[\Lambda]$ in $\mathbb{Q}_p^n$. Example for $\mathrm{SL}_2(\mathbb{Q}_2)$: the Bruhat–Tits-tree



**Good to know**:
- Links are spherical buildings for $\mathrm{SL}_n(\mathbb{F}_p)$
- Coxeter type A
- Type-C analogues for symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$

## Lead question

How do coefficients of the **Ehrhart polynomial** for a *fixed* lattice polytope $P$ vary as one *varies* the ambient lattice?

## Lead ideas

1. Interpret Ehrhart coeffients as labels on vertices of affine buildings
2. Represent Hecke operators as vertex averaging operators on buildings
3. Use Hecke series to enumerate Ehrhart coefficients

## Ehrhart theory 101

$\Lambda_0 = \mathbb{Z}^n$ - standard lattice in $\mathbb{R}^n$
$P$ - full-dim. lattice polytope in $\mathbb{R}^n$

**Ehrhart polynomial of $P$ relative to $\Lambda_0 := \mathbb{Z}^n$**

$$E_P^\Lambda(T) := \sum_{\ell=0}^n c_\ell^\Lambda(P) T^n \in \mathbb{Q}[T]$$

*Raison d'être*: $E_P^\Lambda(m) = \# mP \cap \mathbb{Z}^n$

**Fun facts about Ehrhart polynomials**:

(A) Constant term $c_0^\Lambda(P) = 1$.
(B) Leading term $c_n^\Lambda(P) = \mathrm{vol}(P)$.
(C) $c_{n-1}^\Lambda(P) \approx$ sum of volumes of faces of $P$.
(D) Other coefficients $c_\ell(P) \in \mathbb{Q}$ - mystery?!?

## Ehrhart–Hecke zeta functions of type A

**Definition (AMV)**

*Type-A Ehrhart–Hecke zeta function*

$$\mathcal{Z}_{n,\ell}^{\mathrm{A}}(s) := \sum_{\mathbb{Z}_p^n \le \Lambda \le \mathbb{Q}_p^n} \frac{c_\ell^\Lambda(P)}{c_\ell^{\Lambda_0}(P)} |\Lambda : \mathbb{Z}_p^n|^{-s}$$

**Theorem (AMV)**

With $\zeta_p(s) = \frac{1}{1-p^{-s}}$ (Riemann):

$$\mathcal{Z}_{n,\ell}^{\mathrm{A}}(s) = \zeta_p(s-\ell) \prod_{k=1}^{n-1} \zeta_p(s-k)$$

Note independence of $P$!
Illustrated in the example →.

**Definition (AMV)**

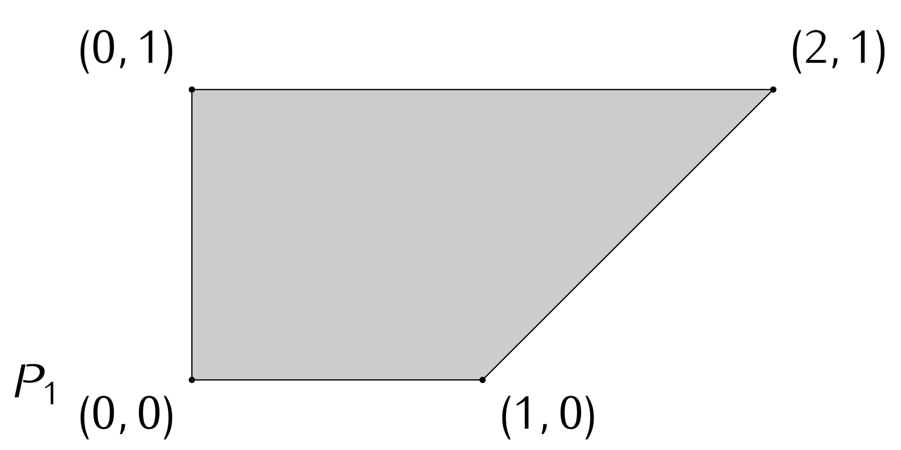
*Type-C Ehrhart–Hecke zeta function*

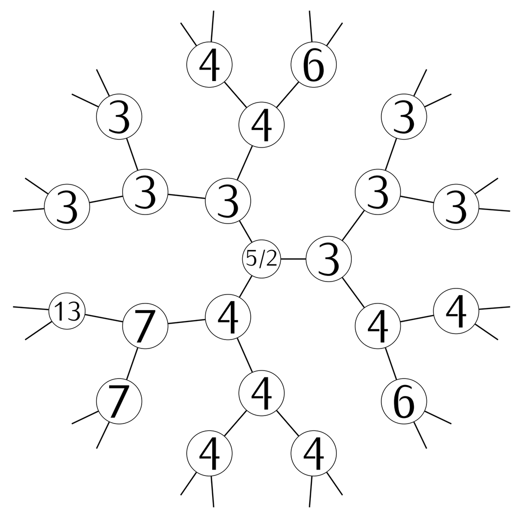
$$\mathcal{Z}_{n,\ell}^{\mathrm{C}}(s) := \sum_{\mathbb{Z}_p^n \le \Lambda \le \mathbb{Q}_p^n} \frac{c_\ell^\Lambda(P)}{c_\ell^{\Lambda_0}(P)} |\Lambda : \mathbb{Z}_p^n|^{-s}$$

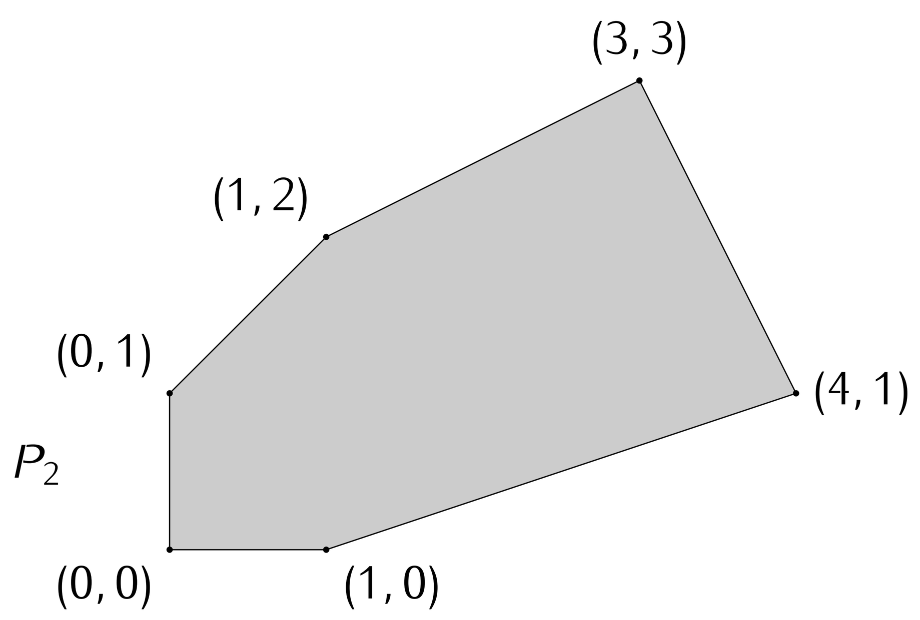
*Formula for $\mathcal{Z}_{n,\ell}^{\mathrm{C}}(s)$? That's what this project is about! Find the one for $n=3$ and (simultaneously) all $\ell$ at the bottom of the page.* ↓

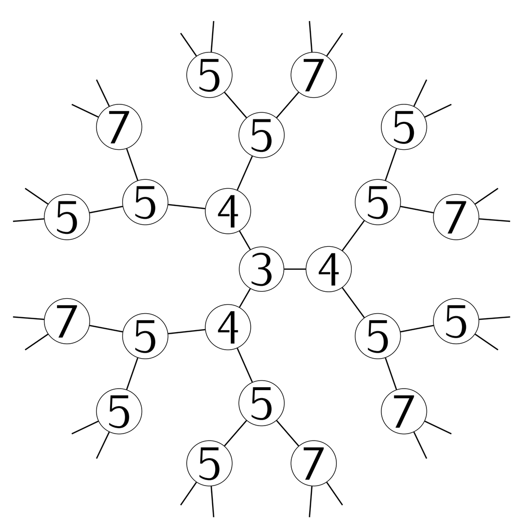
## Example – $n = 2$, $\ell = 1$

Consider the linear terms $c_1^\Lambda(P_i)$ of the Ehrhart polynomials $E_{P_i}^\Lambda(t) = 1 + c_1^\Lambda(P_i)t + t^2$ of two polytopes $P_1$ and $P_2$ as a function of the overlattices $\Lambda \supseteq \mathbb{Z}_p^2$ of finite index.




Note: weight functions depend on $P_i$, **but** sums over the concentric circles around the root (labeled 5/2 and 3), normalized by this label, do not. Surprise?!

## Symplectic Hecke theory (the small print)

- $G_n = \mathrm{GSp}_{2n}(\mathbb{Q}_p)$
- $G_n^+ = \mathrm{GSp}_{2n}(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p)$
- $\Gamma_n = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$
- $\mathcal{H}_n^{\mathrm{C}} = \mathcal{H}(G_n^+, \Gamma_n)$ – spherical Hecke algebra
- $J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{Z})$
- $D_n^{\mathrm{C}}(p^\alpha) = \{A \in G_n^+ \mid AJA^t = p^\alpha J\}$
- $T_n^{\mathrm{C}}(p^\alpha) = \sum_{\Gamma_n \backslash D_n^{\mathrm{C}}(p^\alpha)/\Gamma_n} \Gamma_n g \Gamma_n \in \mathcal{H}_n^{\mathrm{C}}$
- $W$ – type-C Weyl group, aka signed permutations
- $\Omega : \mathcal{H}_p^{\mathrm{C}} \xrightarrow{\sim} \mathbb{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]^W$ – Satake isom.

**Enjoy the whole story**:
C. Alfes, J. Maglione, C. Voll, *Ehrhart polynomials, Hecke series, and affine buildings*, FPSAC 2024

TRR 358

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## Hermite–Smith series

Two sets of invariants of a lattice $\Lambda \le \mathbb{Z}_p^n$:

**Smith** parameters: $\mu(\Lambda) = (\mu_1, \dots, \mu_n)$ | $\mathbb{Z}_p^n/\Lambda \cong \bigoplus_{i=1}^n \mathbb{Z}/(p^{\mu_i})$

**Hermite** parameters: $\delta(\Lambda) = (\delta_1, \dots, \delta_n)$ | $\Lambda \longleftrightarrow \begin{pmatrix} p^{\delta_1} & * & * & * \\ & p^{\delta_2} & * & * \\ & & \ddots & \vdots \\ & & & p^{\delta_n} \end{pmatrix}$

**Definition (AMV)**

The *Hermite–Smith series* enumerates $p$-adic lattices simultaneously by Hermite- and Smith-parameters:

$$\mathrm{HS}_n(s) := \sum_{\Lambda \le \mathbb{Z}_p^n \text{ primitive}} \boldsymbol{X}^{\mu(\Lambda)} \boldsymbol{Y}^{\delta(Y)} \in \mathbb{Q}[\![\boldsymbol{X}, \boldsymbol{Y}]\!]$$

Example: $$\mathrm{HS}_{2,p}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1 - X_1^2 Y_1 Y_2}{(1 - X_1Y_1)(1 - pX_1Y_2)(1 - X_2Y_1Y_2)}.$$

**Theorem (AMV)**

Explicit combinatorial formula for all $n \in \mathbb{N}$ for
$\mathrm{HS}_{n,p}(X_1, \dots, X_n, 1, \dots, 1, Y_n)$
as rational function in $p$ (!) and $\boldsymbol{X}$ and $Y$

## Symplectic Hecke series, Satake generating functions, and Hermite–Smith series

Consider the *symplectic Hecke series*

$$\sum_{\alpha \ge 0} T_n^{\mathrm{C}}(p^\alpha) X^\alpha \in \mathcal{H}_p^{\mathrm{C}}[\![X]\!].$$

Applying the Satake isomorphism $\Omega$ to its coefficients, we obtain

**Definition (local Satake generating function):**

$$R_n(x_0, \dots, x_n, X) := (1 - x_0X)(1 - x_0x_1 \dots x_nX) \sum_{\alpha \ge 0} \Omega(T_n^{\mathrm{C}}(p^\alpha)) X^\alpha \in \mathbb{C}[\boldsymbol{x}^{\pm 1}][\![X]\!]$$

**Theorem (Andrianov)**

The Hermite–Smith series specializes to the local Satake generating function:

$$R_n(x_0, x_1, \dots, x_n, X) = \mathrm{HS}_{n,p}\left( \left(p^{\binom{i+1}{2}} x_0 X\right)_{i=1}^n, \left(p^{-j} x_j\right)_{j=1}^n \right)$$

**Theorem (AMV)**

$$\mathrm{HS}_{n,p}\left( \left(p^{\binom{i}{2}+\ell-ns}\right)_{i=1}^n, 1^{(n-1)}, p^{-\ell} \right) = \mathcal{Z}_{n,\ell}^{\mathrm{C}}(s)(1 - p^{\ell-s})(1 - p^{\binom{n+1}{2}-s})$$

## An example in dimension 6 – the Ehrhart–Hecke zeta function of type $\mathrm{C}_3$

$$\mathcal{Z}_{3,\ell}^{\mathrm{C}}(s) = \left. \frac{1 + (X^{1+\ell} + X^4)Y - (X^{7+\ell} + 2X^{6+\ell} + 2X^{4+\ell} + X^{3+\ell})Y^2 + (X^{6+2\ell} + X^{9+\ell})Y^3 + X^{10+2\ell}Y^4}{(1 - X^3Y)(1 - X^5Y)(1 - X^6Y)(1 - X^\ell Y)(1 - X^{2+\ell}Y)(1 - X^{3+\ell}Y)} \right|_{X \to p, Y \to p^{-3s}} \quad \text{for } \ell \in \{0, 1, \dots, 6\}$$