Colored Permutation Statistics by Conjugacy Class

Kevin Liu

Joint work with Jesse Campion Loth, Michael Levet, Sheila Sundaram, and Mei Yin

University of Washington Department of Mathematics

Colored permutation groups

The colored permutation group $\mathfrak{S}_{n,r}$ is the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n$, which is the semidirect product $\mathbb{Z}_r^n \rtimes \mathfrak{S}_n$ formed from the permutation action of $\mathfrak{S}_n$ on $\mathbb{Z}_r^n$.

$\mathfrak{S}_{n,r}$ can be viewed as a subgroup of $\mathfrak{S}_{nr}$. Consider r copies of the integers $\{1, 2, \dots, n\}$, each colored by an element in $\mathbb{Z}_r$:

$$\left\{ i^{[c]} : i \in \{1, 2, \dots, n\}, [c] \in \mathbb{Z}_r \right\}.$$

The colored permutation group $\mathfrak{S}_{n,r}$ can be viewed as the permutations on this set satisfying the condition that

$$\text{if } \omega\left(i^{[0]}\right) = j^{[c]}, \text{ then } \omega\left(i^{[h]}\right) = j^{[c]+[h]} \text{ for all } [h] \in \mathbb{Z}_r.$$

In this setting, the group operation is function composition.

Examples

A colored permutation $\omega \in \mathfrak{S}_{5,4}$ can be expressed in two-line and one-line notations by specifying the images of the elements with color $[0]$:

$$\omega = \begin{bmatrix} 1^{[0]} & 2^{[0]} & 3^{[0]} & 4^{[0]} & 5^{[0]} \\ 4^{[1]} & 5^{[3]} & 1^{[3]} & 3^{[1]} & 2^{[1]} \end{bmatrix} = \begin{bmatrix} 4^{[1]} & 5^{[3]} & 1^{[3]} & 3^{[1]} & 2^{[1]} \end{bmatrix}.$$

A colored permutation can also be expressed in the two-line and one-line cycle notations:

$$\omega = \begin{pmatrix} 1^{[0]} & 4^{[0]} & 3^{[0]} \\ 4^{[1]} & 3^{[1]} & 1^{[3]} \end{pmatrix} \begin{pmatrix} 2^{[0]} & 5^{[0]} \\ 5^{[3]} & 2^{[1]} \end{pmatrix} = \begin{pmatrix} 4^{[1]} & 3^{[1]} & 1^{[3]} \end{pmatrix} \begin{pmatrix} 5^{[3]} & 2^{[1]} \end{pmatrix}.$$

Special cases

1. The symmetric group $\mathfrak{S}_n$ is isomorphic to $\mathfrak{S}_{n,1} = \mathbb{Z}_1 \wr \mathfrak{S}_n$ through identifying the integer i with $i^{[0]}$.
2. The signed symmetric group B_n consists of bijections on $\{\pm 1, \pm 2, \dots \pm n\}$ satisfying $\omega(-i) = -\omega(i)$ with group operation given by function composition. This is isomorphic to $\mathfrak{S}_{n,2} = \mathbb{Z}_2 \wr \mathfrak{S}_n$ by identifying i with $i^{[0]}$ and $-i$ with $i^{[1]}$ for all $i \in \{1, 2, \dots, n\}$.

Conjugacy classes of colored permutation groups

Express $\omega \in \mathfrak{S}_{n,r}$ in cycle notation as $\sigma_1 \sigma_2 \dots \sigma_\ell$. The *color* of σ_i is the sum of the colors that appear in the cycle (as an element in $\mathbb{Z}_r$).

The *cycle type* of ω is the r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{[0]}, \lambda^{[1]}, \dots, \lambda^{[r-1]})$ where $\lambda^{[c]}$ records cycle lengths for the cycles of color $[c]$ in ω.

Fact. Two elements in $\mathfrak{S}_{n,r}$ are in the same conjugacy class if and only if they share the same cycle type.

Notation

1. For any partition $\lambda \vdash n$, C_λ denotes the permutations in $\mathfrak{S}_n$ with cycle type λ.
2. For any partition $\lambda \vdash n$, the number of parts in λ with length i is denoted $a_i(\lambda)$.
3. For any r-tuple of partitions $\boldsymbol{\lambda} \vdash_r n$, $C_{\boldsymbol{\lambda}}$ denotes the permutations in $\mathfrak{S}_{n,r}$ with cycle type $\boldsymbol{\lambda}$.

Means of some statistics on conjugacy classes

Direct calculation shows that the mean (or first moment) of various statistics on conjugacy classes C_λ of the symmetric group $\mathfrak{S}_n$ depends on n and the number of cycles of "short" length.

Statistic	Mean on C_λ
$\mathrm{inv}(\omega) = \#\{(i,j) : 1 \le i < j \le n, \omega(i) > \omega(j)\}$	$\frac{3n^2 - n + 2a_2(\lambda) - a_1(\lambda)^2 + a_1(\lambda) - 2na_1(\lambda)}{12}$
$\mathrm{des}(\omega) = \#\{i : 1 \le i \le n-1, \omega(i) > \omega(i+1)\}$	$\frac{n^2 - n + 2a_2(\lambda) - a_1(\lambda)^2 + a_1(\lambda)}{2n}$
$\mathrm{exc}(\omega) = \#\{i : 1 \le i \le n-1, \omega(i) > i\}$	$\frac{n - a_1(\lambda)}{2}$
$\mathrm{cval}(\omega) = \#\{i : 1 \le i \le n, \omega^{-1}(i) > i < \omega(i)\}$	$\frac{n - a_1(\lambda) + a_2(\lambda)}{3}$

Constraints and indicator functions

A *constraint* is a set of ordered pairs of the form

$$K = \left\{ \left(i_h^{[0]}, j_h^{[c_h]} \right) \right\}_{h=1}^{k}.$$

This constraint has a corresponding indicator function $I_K : \mathfrak{S}_{n,r} \to \mathbb{R}$ defined by

$$I_K(\omega) = \begin{cases} 1 & \text{if } \omega\left(i_h^{[0]}\right) = j_h^{[c_h]} \text{ for } h = 1, 2, \dots, k \\ 0 & \text{otherwise.} \end{cases}$$

Moments on conjugacy classes without short cycles

A statistic $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ is *realizable over constraints of size* k if

$$X \in \mathrm{span}_{\mathbb{R}} \{ I_K : |K| \le k \}.$$

Theorem 1 (Campion Loth, Levet, Liu, Sundaram, Yin). Suppose $X : \mathfrak{S}_{n,r} \to \mathbb{R}$ is realizable over a constraint set of size k. For any $k \ge 1$, the m-th moment of X coincides on all conjugacy classes $C_{\boldsymbol{\lambda}}$ with no cycles of length $1, 2, \dots, km$.

Example

Fix the following ordering:

$$1^{[0]} < 2^{[0]} < 3^{[0]} < \ldots < 1^{[1]} < 2^{[1]} < 3^{[1]} < \ldots < 1^{[r-1]} < 2^{[r-1]} < 3^{[r-1]} < \ldots.$$

Using this ordering, define the descent statistic on $\mathfrak{S}_{n,r}$:

$$\mathrm{des}(\omega) = \#\{i : 1 \le i \le n, \omega(i^{[0]}) > \omega((i+1)^{[0]})\},$$

where by convention, $(n+1)^{[0]}$ is a fixed point of ω.

This statistic is realizable over a constraint set of size 2:

$$\mathrm{des} = \sum_{i=1}^{n-1} \sum_{j_1^{[c_1]} < j_2^{[c_2]}} I_{\{(i^{[0]}, j_2^{[c_2]}), ((i+1)^{[0]}, j_1^{[c_1]})\}} + \sum_{j=1}^{n} \sum_{c=1}^{r-1} I_{\{(n^{[0]}, j^{[c]})\}}.$$

Consequently, its m-th moment coincides on all $C_{\boldsymbol{\lambda}}$ without cycles of length $1, 2, \dots, 2m$.

Descents in the signed symmetric group

Using the usual integer ordering on $\{\pm 1, \pm 2, \dots, \pm n\}$, define the type-B descent statistic on B_n:

$$\mathrm{des}_B(\omega) = \#\{i : 0 \le 1 \le n-1 : \omega(i) > \omega(i+1),$$

where by convention, 0 is a fixed point of ω.

Distribution on conjugacy classes

For nonnegative integers j and i, define

$$N(j, 2i) = \frac{1}{2i} \sum_{\substack{d \mid i \\ d \text{ odd}}} \mu(d) \left(j^{i/d} - 1 \right),$$

where $\mu(d)$ is the number-theoretic Möbius function.

Theorem 2 (Campion Loth, Levet, Liu, Sundaram, Yin). For $C_{\boldsymbol{\lambda}} \subseteq B_n$ where $\boldsymbol{\lambda} \neq ((1^n), \emptyset)$, define $A_{C_{\boldsymbol{\lambda}}}(t) = \sum_{\omega \in C_{\boldsymbol{\lambda}}} t^{\mathrm{des}_B(\omega)+1}$. Then $A_{C_{\boldsymbol{\lambda}}}(t)/(1-t)^n$ is given by

$$\sum_{j \ge 1} t^j \binom{N(2j-1, 2) + a_1(\lambda^{[0]})}{a_1(\lambda^{[0]})} \prod_{i \ge 2} \binom{N(2j-1, 2i) + a_i(\lambda^{[0]}) - 1}{a_i(\lambda^{[0]})} \prod_{i \ge 1} \binom{N(2j-1, 2i)}{a_i(\lambda^{[1]})}.$$

Consequently, the number of permutations in $C_{\boldsymbol{\lambda}}$ with $d-1$ descents is

$$\sum_{j=1}^{d} (-1)^{d-j} \binom{n+1}{d-j} \binom{N(2j-1, 2) + a_1(\lambda^{[0]})}{a_1(\lambda^{[0]})} \prod_{i \ge 2} \binom{N(2j-1, 2i) + a_i(\lambda^{[0]}) - 1}{a_i(\lambda^{[0]})} \prod_{i \ge 1} \binom{N(2j-1, 2i)}{a_i(\lambda^{[1]})}.$$

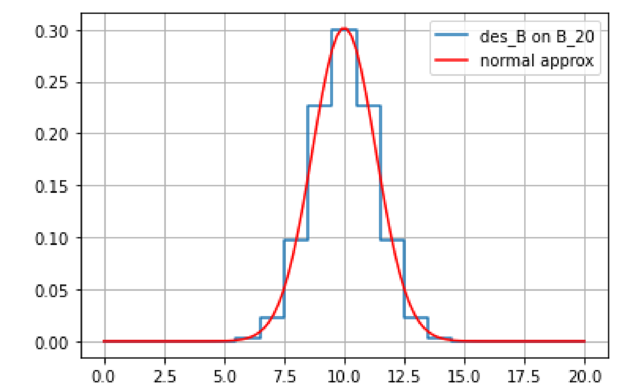
Asymptotic result

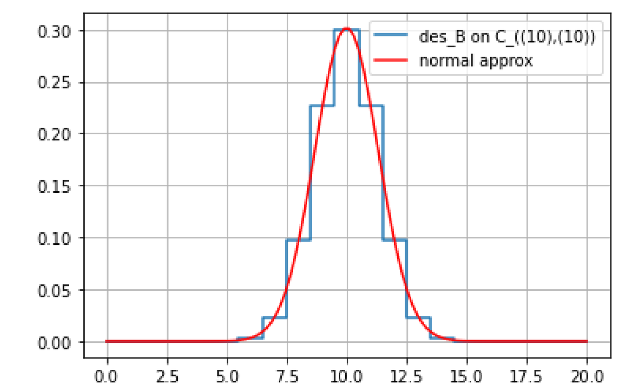
Theorem 3 (Campion Loth, Levet, Liu, Sundaram, Yin). For any $C_{\boldsymbol{\lambda}} \subseteq B_n$ with no cycles of length $1, 2, \dots, 2m$, the m-th moment of des_B on $C_{\boldsymbol{\lambda}}$ is equal to the m-th moment of des_B on B_n.

Corollary 4 (Campion Loth, Levet, Liu, Sundaram, Yin). For every $n \ge 1$, let $C_{\boldsymbol{\lambda}_n} \subseteq B_n$ be a conjugacy class such that the following holds for all $i \in \mathbb{N}$:

$$\lim_{n \to \infty} a_i(\lambda^{[0]}) = \lim_{n \to \infty} a_i(\lambda^{[1]}) = 0.$$

Then des_B has the same asymptotic distribution on $C_{\boldsymbol{\lambda}_n}$ and B_n, and hence is normal.





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