Two-Row Set-Valued Tableaux: Catalan^{+k} Combinatorics

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Usual Representation

Filling of Ferrers diagram for $\lambda \vdash n$ with k *extra elements*

$S =$

1	2	7	8
3	4, 5	11	13
6, 9, 10	12	14, 15	16

$\mathrm{SYT}^{+k}(\lambda)$ = set of set-valued SYT of λ with k extra elts

Set-Valued Descents

$i \in [n+k]$ is a *descent* of $S \in \mathrm{SYT}^{+k}(\lambda)$ if:

- $i, i+1 \in \lambda^{(j)}$ and i below $i+1$, or
- i is an extra element

$\mathrm{D}^{+k}(S)$ = set of set-valued descents

$$\mathrm{comaj}^{+k}(S) := \sum_{i \in \mathrm{D}^{+k}(S)} (n+k-i)$$

$$\mathrm{D}^{+3}(S) = \{6, 12\} \cup \{5, 9, 10, 15\}$$
$$\mathrm{comaj}^{+3}(S) = 10+4+11+7+6+1 = 39$$

Comaj Generating Function

Enumeration is very difficult in general. Special case of rectangular shape with $k = 1$ is known:

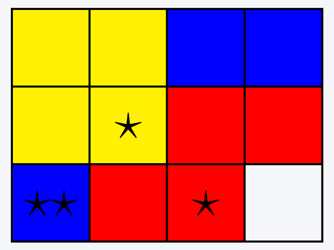
Theorem (Hopkins – Lazar – Linusson [2]):
For all positive integers a, b, we have

$$\sum_{S \in \mathrm{SYT}^{+1}(a\times b)} q^{\mathrm{comaj}^{+1}(S)} = \frac{[a]_q[b]_q}{[a+b]_q}[ab+1]_q! \prod_{i=0}^{a-1} \frac{[i]_q!}{[b+i]_q!}.$$

Best known general results when $q = 1$ are determinental formulas of Anderson–Chen–Tarrasca [1].

Alternative Representation

A filtration $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \lambda^{(k)} \subseteq \lambda^{(k+1)} = \lambda$



1	2	6	7
3	4	8	10
5	9	11	12

along with an outer corner for each $\lambda^{(i)}$ and an ordinary SYT of shape λ

Catalan Combinatorics

In two-row case, fixing the total number of elements yields good behavior:

Theorem (Lazar – Linusson):
For fixed i, n with $0 \le i \le n$,

$$\sum_{2b+k-i=n} \left|\mathrm{SYT}^{+k}((b, b-i))\right| = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

In particular, when $i = 0$

$$\sum_{2b+k=n} \left|\mathrm{SYT}^{+k}((b,b))\right| = \mathrm{Cat}(n-1).$$

Further refinements to the Narayana and Kreweras numbers also exist.

For example, when $n = 4, i = 0$, we have

1	
2, 3, 4	

1, 2	
3, 4	

1, 2, 3
4

1	2
3	4

1	3
2	4

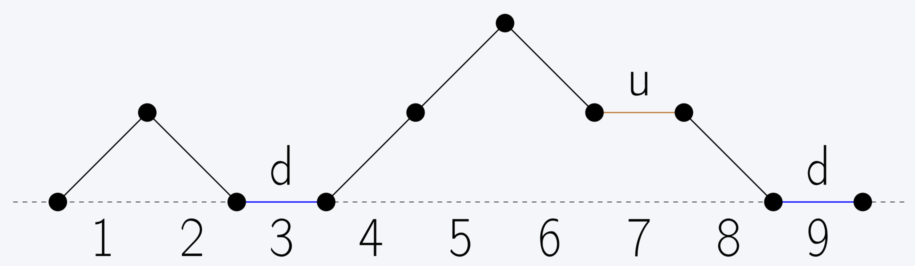
Bijections to Other Catalan Models: 321-Avoiding Permutations and Motzkinlike paths

1, 2	3, 4, 6	7	10
5, 8	9	11, 12	13, 14

$\overset{\alpha}{\mapsto}$ $\overline{1}\ 5\ 8\ \overline{2}\ \overline{3}\ \overline{4}\ 9\ \overline{6}\ 11\ 12\ \overline{7}\ 13\ \overline{10}$

1	4	5, 7
2, 3	6	8, 9

$\overset{\beta}{\mapsto}$



More Lattice Paths

Theorem (Lazar–Linusson):
Consider the following restrictions on bicolored Motzkin paths from $(0,0)$ to $(n, 0)$:

1. No u steps on $y = 0$
2. No d steps before the first down-step.

We have

- $|\mathrm{Motz}(n)| = \mathrm{Cat}(n+1)$
- $|\mathrm{Motz}_{\{1\}}(n)| = |\mathrm{Motz}_{\{2\}}(n)| = \mathrm{Cat}(n)$
- $|\mathrm{Motz}_{\{1,2\}}(n)| = \mathrm{Cat}(n-1)$,

where $\mathrm{Motz}_X(n)$ is the set of bic. Motzkin paths subject to the conditions in $X \subseteq \{1, 2\}$.

Expected # Columns?

Conjecture: Fix $n \ge 3$. Sampling uniformly at random from $\bigsqcup_{2\times b+k=n} \mathrm{SYT}^{+k}((b,b))$, the expected value of b is:

$$\frac{(n-2)(n+3)}{2(2n-3)}$$

q-Catalan?

$$\widetilde{\mathrm{Cat}}_n(q) := \sum_{2b+k=n+1} \left(\sum_{S \in \mathrm{SYT}^{+k}(2\times b)} q^{\mathrm{comaj}^{+k}(S)} \right)$$

n	$\widetilde{\mathrm{Cat}}_n(q)$
1	1
2	$q+1$
3	q^3+2q^2+q+1
4	$q^6+2q^5+3q^4+3q^3+2q^2+2q+1$
5	$q^{10}+2q^9+3q^8+7q^7+6q^6+5q^5+6q^4+7q^3+3q^2+q+1$

Question:
These are *not* any of the usual q-Catalan numbers. Is there a better formula?

[1] Dave Anderson, Linda Chen, and Nicola Tarasca, *K-classes of Brill-Noether Loci and a Determinantal Formula*, Int. Math. Res. Not. IMRN (2022), no. 16, 12653–12698. MR 4466009

[2] Sam Hopkins, Alexander Lazar, and Svante Linusson, *On the q-enumeration of barely set-valued tableaux and plane partitions*, European J. Combin. **113** (2023), Paper No. 103760, 29pp. MR 4611147

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