Jack Derangements

Nathan Lindzey

Department of Computer Science
Technion (Israel Institute of Technology), Israel

Abstract

For each $\lambda \vdash n$ we give a simple combinatorial expression for the sum of the Jack character θ_α^λ over the integer partitions of n with no singleton parts. For $\alpha = 1, 2$ this gives closed forms for the eigenvalues of the permutation and perfect matching derangement graphs, resolving an open question in algebraic graph theory. A byproduct of the latter is a simple combinatorial formula for the immanants of the matrix $J - I$ where J is the all-ones matrix, which might be of independent interest. Our proofs center around a Jack analogue of a hook product related to Cayley's Ω–process of invariant theory that we call *the principal lower hook product*.

Jack Characters

For any $\alpha \in \mathbb{R}$, the *(integral form) Jack polynomals* J_λ are defined as the unique family satisfying the following relations:

- Orthogonality: $\langle J_\lambda, J_\mu \rangle_\alpha = 0$ whenever $\lambda \neq \mu$.
- Triangularity: $J_\lambda = \sum_{\mu \trianglelefteq \lambda} c_{\lambda\mu} m_\mu$
- Normalization: $[m_{1^n}] J_\lambda = n!$.

Specializing α recovers classical families of symmetric functions:

- $\alpha = 1 \longrightarrow$ (integral form) Schur polynomials S_λ.
- $\alpha = 2 \longrightarrow$ (integral form) Zonal polynomials Z_λ.

The *Jack characters* θ_α^λ are the coefficients of the power sum expansion of the J_λ's:

$$J_\lambda = \sum_{\mu \vdash n} \theta_\alpha^\lambda(\mu) p_\mu \quad \text{for all } \lambda \vdash n.$$

- $\alpha = 1 \longrightarrow$ (normalized) irreducible characters of S_n.
- $\alpha = 2 \longrightarrow$ (normalized) zonal spherical functions of S_{2n}/B_n.

Jack Derangement Character Sums

For any $\lambda \vdash n$ and $\alpha \in \mathbb{R}$, we define

$$\eta_\alpha^\lambda := \sum_{\substack{\mu \vdash n \\ \mu \text{ has no singleton parts}}} \theta_\alpha^\lambda(\mu).$$

to be the λ-*Jack derangement sum*.

λ-Colored Permutations

For a given integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$, define

- $\{1, 2, \ldots, \lambda_1\}$ to be the set of *symbols*, and
- for each symbol i, define a *color list* $L_i := \{1, 2, \ldots, \lambda_i^\top\}$.

We define a λ-*colored permutation* (c, σ) to be

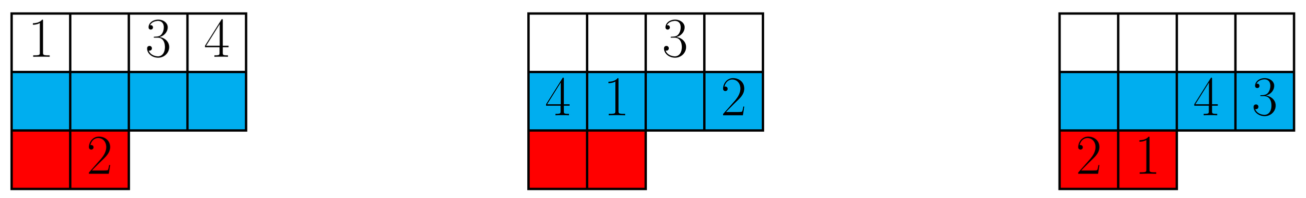
- an assignment of colors $c = c_1, c_2, \ldots, c_{\lambda_1}$ such that $c_i \in L_i$,
- and a permutation $\sigma \in \text{Sym}(\{1, 2, \ldots, \lambda_1\})$ such that

$$\sigma(i) = j \Rightarrow c_i = c_j,$$

i.e., each cycle of the permutation is *monochromatic*.

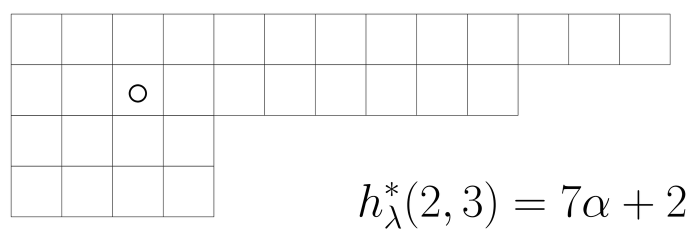
<u>Ex.</u> $(4, 4, 2) \vdash 10$. $L_1 = L_2 =$ ▭■■ and $L_3 = L_4 =$ ▭■.

(1)(2)(3)(4) (1, 2, 4)(3) (1, 2)(3, 4)



Let $h_\lambda^*(i, j) := a_\lambda(i, j)\alpha + l_\lambda(i, j) + 1$ be the *lower hook length*.

<u>Ex.</u>



$h_\lambda^*(2, 3) = 7\alpha + 2 + 1$

Theorem 1 *Let $\lambda \vdash n$ and $\alpha = 1$. The number of λ-colored permutations is*

$$h_\lambda(1, 1) h_\lambda(1, 2) \cdots h_\lambda(1, \lambda_1).$$

(A similar result holds for $\alpha = 2$ and λ-colored perfect matchings.)

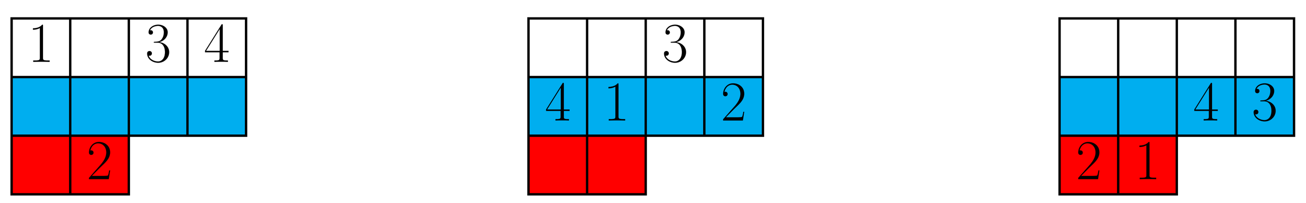
λ-Colored Derangements

A λ-*colored derangement* is a λ-colored permutation (c, σ) s.t.

$$\sigma(i) = i \Rightarrow c_i \neq 1.$$

<u>Ex.</u> $(4, 4, 2) \vdash 10$. $L_1 = L_2 =$ ▭■■ and $L_3 = L_4 =$ ▭■.

(1)(2)(3)(4) (1, 2, 4)(3) (1, 2)(3, 4)



Jack Derangement Numbers

Let d_k^λ denote the number of λ-colored derangements with exactly k cycles. For any $\lambda \vdash$ and $\alpha \in \mathbb{R}$, we define

$$D_\alpha^\lambda := \sum_{k=1}^{\lambda_1} d_k^\lambda \alpha^{\lambda_1 - k}$$

to be the λ-*Jack derangement number*.

Theorem 2 (Main Result I) *For any $\lambda \vdash n$ and $\alpha \in \mathbb{R}$, we have*

$$\eta_\alpha^\lambda = (-1)^{n - \lambda_1} D_\alpha^\lambda$$

Derangement Graphs

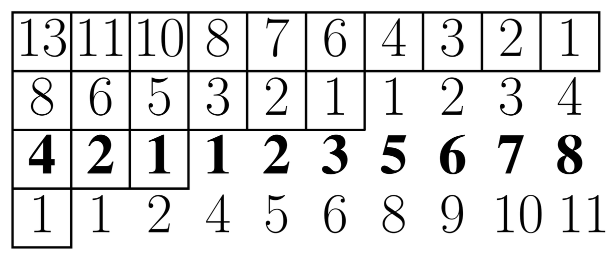
Theorem 2 gives new formulas for the eigenvalues of *derangement graphs*.

- The ***permutation derangement graph*** is the Cayley graph $\text{Cay}(S_n, D_n)$ on S_n generated by its derangements D_n. Its eigenvalues are $\{\eta_1^\lambda : \lambda \vdash n\}$.
- The ***perfect matching derangement graph*** is the graph over the perfect matchings of the complete graph K_{2n} defined such that two perfect matchings are adjacent if they share no edges. Its eigenvalues are $\{\eta_2^\lambda : \lambda \vdash n\}$.

Using the umbral calculus, we obtain closed forms for their eigenvalues.

- Define $p_{m,j} := \Pr_{\sigma \in S_m}[\sigma \text{ has } j \text{ fixed points}]$.
- We define $H_i^+(\lambda)$ to be the *extended ith principal hook product*.

<u>Ex.</u> $\lambda = (10, 6, 3, 1) \vdash 20$, $H_3^+(\lambda) = 80640 = 2 \cdot 8!$.



13	11	10	8	7	6	4	3	2	1
8	6	5	3	2	1	1	2	3	4
4	**2**	**1**	**1**	**2**	**3**	**5**	**6**	**7**	**8**
1	1	2	4	5	6	8	9	10	11

Theorem 3 (Main Result II) *Let $\lambda = (a_1, \ldots, a_d \mid b_1, \ldots, b_d) \vdash n$. Then*

$$\eta_1^\lambda = (-1)^n \sum_{i \leq \lambda_i + 1} (-1)^{\lambda_i} p_{\lambda_1, a_1 - a_i} H_i^+(\lambda).$$

(A similar result holds for $\alpha = 2$.)

Immanants

Let f^λ denote the number of standard Young tableaux of shape λ.

Theorem 4 *Let $\lambda \vdash n$. Then* $\text{Imm}_\lambda(J - I) = f^\lambda D_1^\lambda$.