Regular Schur labeled skew shape posets and their 0-Hecke modules

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1. Preliminaries

Let n be a positive integer.

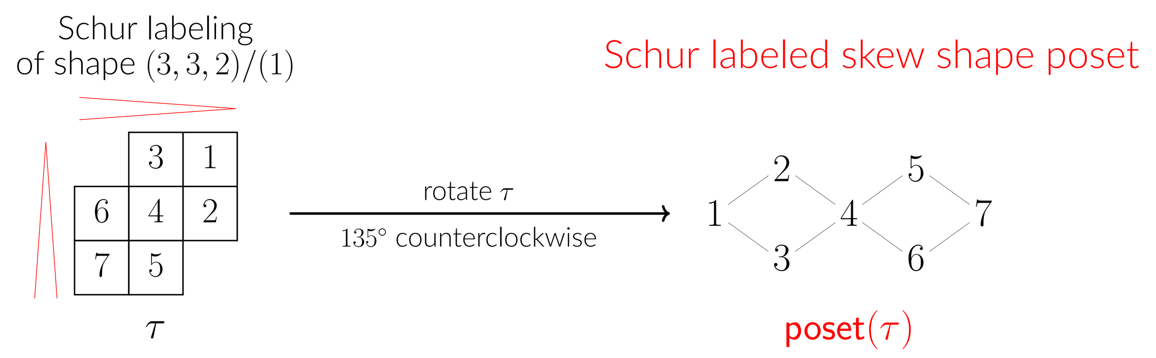
- P_n := the set of all posets on $[n]$, where $[n] = \{1, 2, \ldots, n\}$
- QSym := the ring of quasisymmetric functions
- F_α := the fundamental quasisymmetric function attached to $\alpha \models n$

For $P \in \mathsf{P}_n$, define

- $\Sigma_L(P) := \{\sigma \in \mathfrak{S}_n \mid i \preceq_P j \Rightarrow \sigma(i) \le \sigma(j)\}$ (= the set of linear extensions of P)
- K_P := the P-partition generating function of P $= \sum_{\sigma \in \Sigma_L(P)} F_{\mathrm{comp}(\mathrm{Des}_L(\sigma))^c}$ (1984, I.Gessel)

1.1. Schur labeled skew shape posets

A Schur labeling of shape λ/μ is a bijective tableau of shape λ/μ s.t. each row decreases from left to right and each column increases from top to bottom.



- SP_n := the set of all Schur labeled skew shape posets in P_n

For $P \in \mathsf{SP}_n$, K_P is a symmetric function.

Stanley's P-partitions conjecture. ([5])

For $P \in \mathsf{P}_n$,
K_P: symmetric function $\Rightarrow$ P: Schur labeled skew shape poset.

1.2. Regular posets

For $\sigma, \rho \in \mathfrak{S}_n$,

- $\mathrm{Des}_L(\sigma) := \{i \in [n-1] \mid i \text{ is right of } i+1 \text{ in } \sigma(1)\sigma(2)\cdots\sigma(n)\}$
- the left weak Bruhat order $\preceq_L$ on $\mathfrak{S}_n$: $\gamma \preceq_L s_i\gamma \Leftrightarrow i \notin \mathrm{Des}_L(\gamma)$
- the left weak Bruhat interval $[\sigma, \rho]_L := \{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$

We adopt the following theorem as the definition of regular posets.

Theorem. 1991, Björner-Wachs ([1])

For $P \in \mathsf{P}_n$,
P: regular poset $\Leftrightarrow$ $\Sigma_L(P)$: left weak Bruhat interval in $\mathfrak{S}_n$.

- RP_n := the set of all regular posets in P_n

1.3. 0-Hecke algebras and the quasisymmetric characteristic

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$-alg. gen. by $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following relations:

$$\pi_i^2 = \pi_i \quad \text{for } i \in [n-1],$$
$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} \quad \text{for } i \in [n-2],$$
$$\pi_i\pi_j = \pi_j\pi_i \quad \text{if } |i-j| \ge 2.$$

Note. For $n \ge 4$, there are infinitely many nonisomorphic indecomposable $H_n(0)$-modules. To be precisely, $H_4(0)$ is tame and $H_n(0)$ is wild for $n > 4$.

'79 Norton classified all irreducible $H_n(0)$-modules $\mathbf{F}_\alpha$ ($\alpha \models n$).

'96 Duchamp-Krob-Leclerc-Thibon introduced the ring isomorphism

$$\mathrm{ch} : \bigoplus_{n\ge 0} \mathcal{G}_0(H_n(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha \quad (\alpha\text{: composition}),$$

called the quasisymmetric characteristic.

1.4. 0-Hecke modules arising from posets

Duchamp-Hivert-Thibon([2]) constructed a *right* $H_n(0)$-module M_P ($P \in \mathsf{P}_n$) s.t. $\mathrm{ch}([M_P]) = K_P$. Here, we consider its left module version.

Definition.

Let $P \in \mathsf{P}_n$. Define M_P to be the $H_n(0)$-module with

- the underlying space: $\mathbb{C}\Sigma_L(P)$
- the $H_n(0)$-action: for $\gamma \in \Sigma_L(P)$ and $i \in [n-1]$,

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin \Sigma_L(P), \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in \Sigma_L(P). \end{cases}$$

Note. 1. $\mathrm{ch}([\mathsf{M}_P]) = \psi(K_P)$, where ψ : QSym $\to$ QSym, $F_\alpha \mapsto F_{\alpha^c}$.

2. If $P \in \mathsf{SP}_n$, then $\mathrm{ch}([\mathsf{M}_P]) = s_{\lambda/\mu}$, where λ/μ is the shape of a Schur labeling τ s.t. $P = \mathsf{poset}(\tau)$.

3. The set $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$ contains all indecomposable summands of $H_n(0)$-modules constructed to give a representation-theoretical interpretation of important quasisymmetric functions ([3]).

1.5. Regular Schur labeled skew shape posets

- RSP_n := $\mathsf{RP}_n \cap \mathsf{SP}_n$

1. Assuming Stanley's P-partitions conjecture hold,

$$\mathsf{RSP}_n = \{P \in \mathsf{RP}_n \mid K_P \text{ is a symmetric function}\}.$$

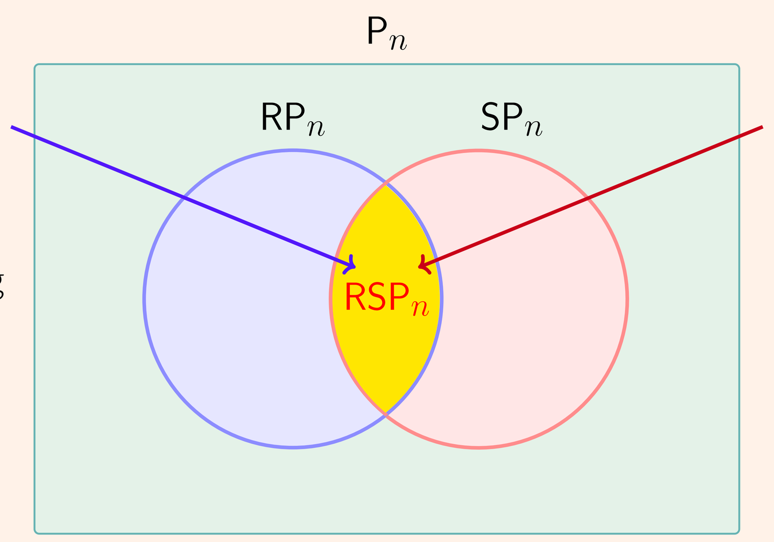
2. For any $P \in \mathsf{SP}_n$, there exist $Q \in \mathsf{RSP}_n$ and $\delta \in \mathfrak{S}_n$ s.t.

$$\Sigma_L(P) = \Sigma_L(Q) \cdot \delta.$$

In addition,

$$\{K_P \mid P \in \mathsf{RSP}_n\} = \{K_P \mid P \in \mathsf{SP}_n\}.$$

Goal



Combinatorial viewpoint

P1. Characterize the posets P in RSP_n using $\Sigma_L(P)$.

P2. Describe the interval $\Sigma_L(P)$ using tableau reading words for $P \in \mathsf{RSP}_n$.

P3. Describe the descent-preserving isomorphism class of $\Sigma_L(P)$ for $P \in \mathsf{RSP}_n$.

Representation theoretical viewpoint

P4. Classify $H_n(0)$-modules M_P's ($P \in \mathsf{RSP}_n$) up to isomorphism.

P5. Explain the Schur-positivity of K_P ($P \in \mathsf{RSP}_n$) from the $H_n(0)$-representation theoretical viewpoint.

2. Combinatorial properties of $P \in \mathsf{RSP}_n$

2.1. A characterization of posets in RSP_n

Theorem 1. ([4])

For $P \in \mathsf{P}_n$,
$P \in \mathsf{RSP}_n \Leftrightarrow \Sigma_L(P)$: dual plactic-closed.

Here, a subset of $\mathfrak{S}_n$ is called dual plactic-closed if it can be written as the union of some dual Knuth equivalence classes.

Example 1. Let $P := {}^{1}\!\searrow_{2}\ 3 \in \mathsf{P}_3$.

(i) $\Sigma_L(P) = \{213, 312, 321\} = [213, 321]_L$ and $P = \mathsf{poset}\left(\begin{smallmatrix} & 2 & 1 \\ 3 & & \end{smallmatrix}\right)$

$\Rightarrow P \in \mathsf{RSP}_3$

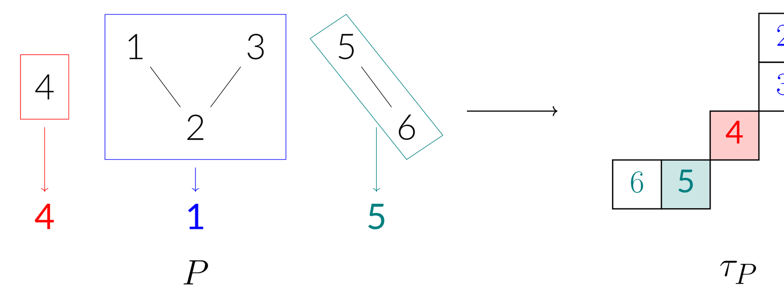
(ii) $213 \overset{\mathrm{RSK}}{\leftrightarrow} \left(\begin{smallmatrix}1&3\\2&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}\right)$, $312 \overset{\mathrm{RSK}}{\leftrightarrow} \left(\begin{smallmatrix}1&2\\3&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}\right)$, $321 \overset{\mathrm{RSK}}{\leftrightarrow} \left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$

$\Rightarrow \{213, 312\}$ and $\{321\}$ are dual Knuth equivalence classes.

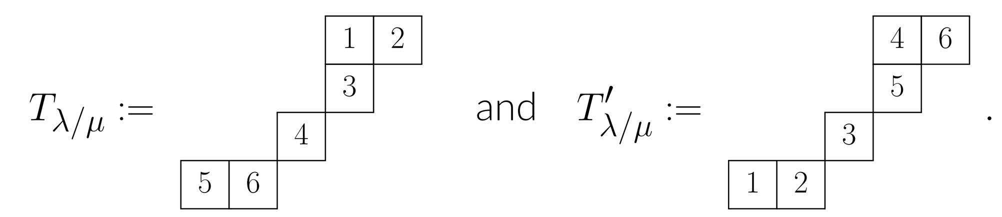
2.2. The left weak Bruhat interval structure

Necessary definitions and notation.

- τ_P is a unique Schur labeling defined by $P = \mathsf{poset}(\tau_P)$ and



- $T_{\lambda/\mu}$, $T'_{\lambda/\mu} \in \mathrm{SYT}(\lambda/\mu)$ are defined as follows:



- read_{τ_P}: $\mathrm{SYT}(\lambda/\mu) \to \mathfrak{S}_n$ is defined by $\mathsf{read}_{\tau_P}(T)(k) = T_{\tau_P^{-1}(k)}$. For instance, $\mathsf{read}_{\tau_P}(T_{\lambda/\mu}) = 213465$.

Theorem 2. ([4])

Let $P \in \mathsf{RSP}_n$ and $\lambda/\mu = \mathrm{sh}(\tau_P)$. Then, $\Sigma_L(P) = \mathsf{read}_{\tau_P}(\mathrm{SYT}(\lambda/\mu))$ and

$$\Sigma_L(P) = [\mathsf{read}_{\tau_P}(T_{\lambda/\mu}), \mathsf{read}_{\tau_P}(T'_{\lambda/\mu})]_L.$$

2.3. The descent-preserving isomorphism class

- $\mathrm{Int}(n)$:= {all left weak Bruhat intervals in $\mathfrak{S}_n$} = $\{\Sigma_L(P) \mid P \in \mathsf{RP}_n\}$

Define an equivalence relation $\overset{D}{\simeq}$ on $\mathrm{Int}(n)$ by $I_1 \overset{D}{\simeq} I_2$ if

$\exists$ a poset iso. $f : (I_1, \preceq_L) \to (I_2, \preceq_L)$ s.t. $\mathrm{Des}_L(\gamma) = \mathrm{Des}_L(f(\gamma)) \quad \forall \gamma \in I_1$.

The main reason for studying $\overset{D}{\simeq}$ is that

if $\Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q)$, then $\mathsf{M}_P \cong \mathsf{M}_Q$ and thus $K_P = K_Q$.

Theorem 3. ([4])

Let $P \in \mathsf{RSP}_n$ and C the equivalence class of $\Sigma_L(P)$ under $\overset{D}{\simeq}$. Then,

$$C = \{\Sigma_L(Q) \mid Q \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_Q) = \mathrm{sh}(\tau_P)\}.$$

Theorem 3 tells us that

1. $\{\Sigma_L(P) \mid P \in \mathsf{RSP}_n\}$ is closed under $\overset{D}{\simeq}$, and
2. for any skew partition λ/μ of size n,

$$C_{\lambda/\mu} := \{\Sigma_L(P) \mid P \in \mathsf{RSP}_n \text{ with } \mathrm{sh}(\tau_P) = \lambda/\mu\}$$

is an equivalence class under $\overset{D}{\simeq}$.

Here, $|C_{\lambda/\mu}| = 1 \Leftrightarrow$ the Young diagram of λ/μ contains no 2×2 boxes.

3. $H_n(0)$-module properties of M_P ($P \in \mathsf{RSP}_n$)

3.1. The classification of M_P's ($P \in \mathsf{RSP}_n$)

Theorem 4. ([4])

For $P, Q \in \mathsf{RSP}_n$,
$\mathsf{M}_P \cong \mathsf{M}_Q \Leftrightarrow \mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q)$.

Sketch of the proof.

($\Leftarrow$) If $\mathrm{sh}(\tau_P) = \mathrm{sh}(\tau_Q)$, by Theorem 3, $\mathsf{M}_P \cong \mathsf{M}_Q$.

($\Rightarrow$) Step 1. Find the projective covers and injective hulls of M_P and M_Q.

Step 2. Show that if $\mathrm{sh}(\tau_P) \neq \mathrm{sh}(\tau_Q)$, then M_P and M_Q have either nonisomorphic projective covers or nonisomorphic injective hulls. □

By Theorem 3, Theorem 4 can be restated as follows: for $P, Q \in \mathsf{RSP}_n$,

$$\mathsf{M}_P \cong \mathsf{M}_Q \quad \Leftrightarrow \quad \Sigma_L(P) \overset{D}{\simeq} \Sigma_L(Q). \qquad (*)$$

3.2. Representation theoretical interpretation of Schur positivity of K_P

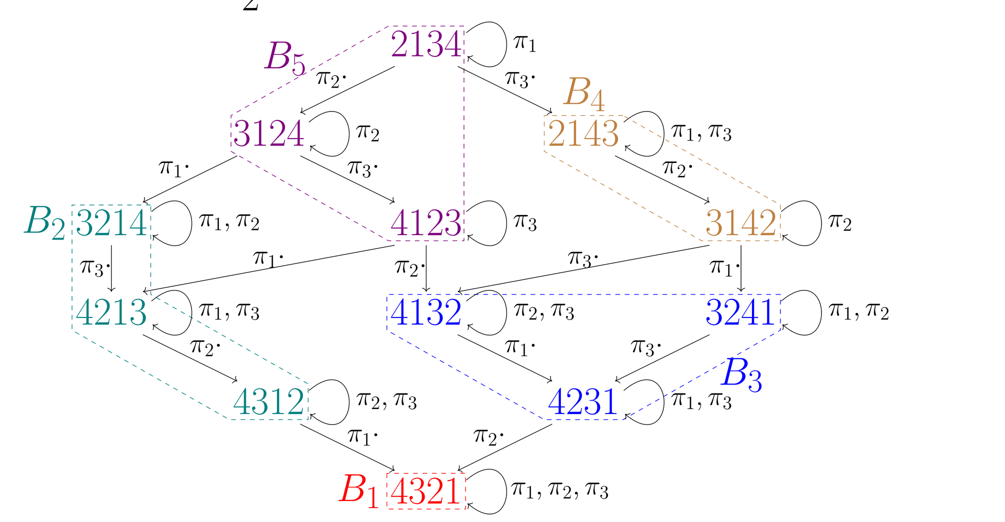
Theorem 5. ([4])

For every $P \in \mathsf{RSP}_n$, there exists a filtration

$$0 =: M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l := \mathsf{M}_P$$

of M_P s.t. for all $1 \le k \le l$, $\mathrm{ch}([M_k/M_{k-1}]) = s_\nu$ for some $\nu \vdash n$.

Example 2. Let $P = {}^{1}\!\searrow_{2}\ 3\ 4 \in \mathsf{RSP}_4$. Then, $\Sigma_L(P) = [2134, 4321]_L$:



For $0 \le k \le 5$, let $\widetilde{B}_k := \bigsqcup_{i \in [k]} B_i$. Then, for $1 \le k \le 5$,

$\mathbb{C}\widetilde{B}_k$ is a submodule of M_P and $\mathrm{ch}([\mathbb{C}\widetilde{B}_k/\mathbb{C}\widetilde{B}_{k-1}])$ is a Schur function.

Remark.

Let M be an $H_n(0)$-module. Even if $\mathrm{ch}([M])$ is Schur-positive, there may not exist a filtration of M that satisfies the property appearing in Theorem 5. For instance, see [4, Example 6.6].

Further avenues

1. Describe the descent-preserving isomorphism class of $I \in \mathrm{Int}(n)$.
2. Classify $\{\mathsf{M}_P \mid P \in \mathsf{SP}_n\}$ and $\{\mathsf{M}_P \mid P \in \mathsf{RP}_n\}$ up to $H_n(0)$-module isomorphism. In particular, we expect that (*) holds for $P, Q \in \mathsf{RP}_n$.

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