Shuffle Theorems and Sandpiles

Michele D'Adderio *Università di Pisa*, Mark Dukes *University College Dublin*, Alessandro Iraci *Università di Pisa*,
Alexander Lazar† *Université Libre de Bruxelles*, Yvan Le Borgne *LaBRI*, Anna Vanden Wyngaerd *Université Libre de Bruxelles*

The Shuffle Theorem

The *shuffle theorem* of Carlsson and Mellit gives a combinatorial expression for coeffs of the symmetric function ∇e_n in terms of a q,t-weighted count of labeled lattice paths:

Theorem (Carlsson–Mellit [1] + Loehr–Remmel [2]):
For all μ, ν compositions with $|\mu|+|\nu|=n$,

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{D\in \mathsf{PF}(\mu;\nu)} q^{\mathsf{area}(D)} t^{\mathsf{pmaj}(D)},$$

where $\mathsf{PF}(\mu;\nu)$ is a certain class of parking functions (viewed as lattice paths).

$\mathsf{area}(D)$ = # whole squares between D and main diagonal.

$\mathsf{pmaj}(D)$ is significantly more complicated.

Graphs

For all μ,ν with $|\mu|+|\nu|=n$, $G_{\mu,\nu}$ is a graph on vertex set $[n]$ with:

- A clique K_{μ_i} for each component $\mu_1,\dots,\mu_{\ell(\mu)}$ of μ.
- An independent set $\bar{K}_{\nu_i}$ for each component $\nu_1,\dots,\nu_{\ell(\nu)}$ of ν.
- All possible edges between these blocks.

We fix a particular labeling of the vertices:

- The vertices of μ_i are $n-(\mu_1+\cdots+\mu_{i-1})$ through $n-(\mu_1+\cdots+\mu_i)+1$.
- The vertices of ν_i are $n-|\mu|-(\nu_1+\cdots+\nu_{i-1})$ through $n-|\mu|-(\nu_1+\cdots+\nu_i)+1$.

Attach a *sink vertex* 0 to each vertex of $G_{\mu,\nu}$ to get $\hat{G}_{\mu,\nu}$.

Running example: $\hat{G}_{(4,3),(2,3)}$

Sandpiles

Sandpile configuration κ on G: assign a nonneg. number of "grains of sand" to each nonsink vertex of G (sink has unlimited sand).

If $\kappa(v)\geq \deg(v)$, v can *topple* and give one grain to each of its neighbors. $\phi_v(\kappa)$ = resulting config. If v can't topple, κ is *stable* at v.

Stable config κ is *recurrent* if, starting from $\phi_{\mathsf{sink}}(\kappa)$, nonsink verts can be ordered so that if we topple in this order the config always stays nonnegative.

Sorted Recurrent Configurations

Sandpiles on $\hat{G}_{\mu,\nu}$ admit an action of the Young subgroup $\mathfrak{S}_\mu\times\mathfrak{S}_\nu$.

Going mod this action $\iff$ sorting the values of κ on each block of vertices corresponding to entries of μ,ν.

Recurrent config κ is *sorted recurrent* if:

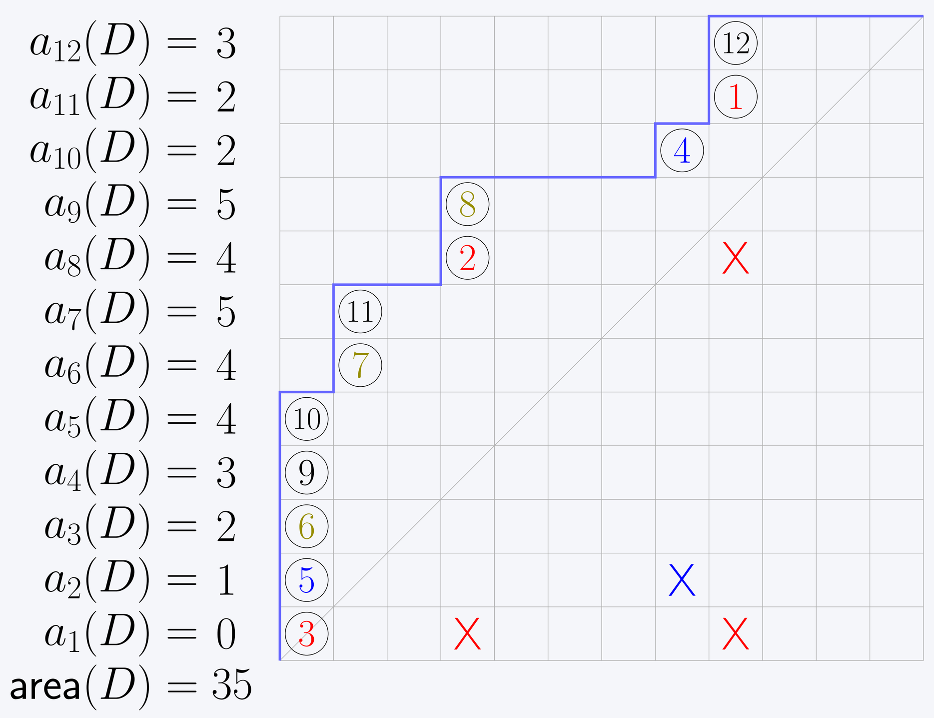
- κ is weakly decreasing inside each clique K_{μ_i},
- κ is weakly increasing inside each independent block $\bar{K}_{\nu_i}$.

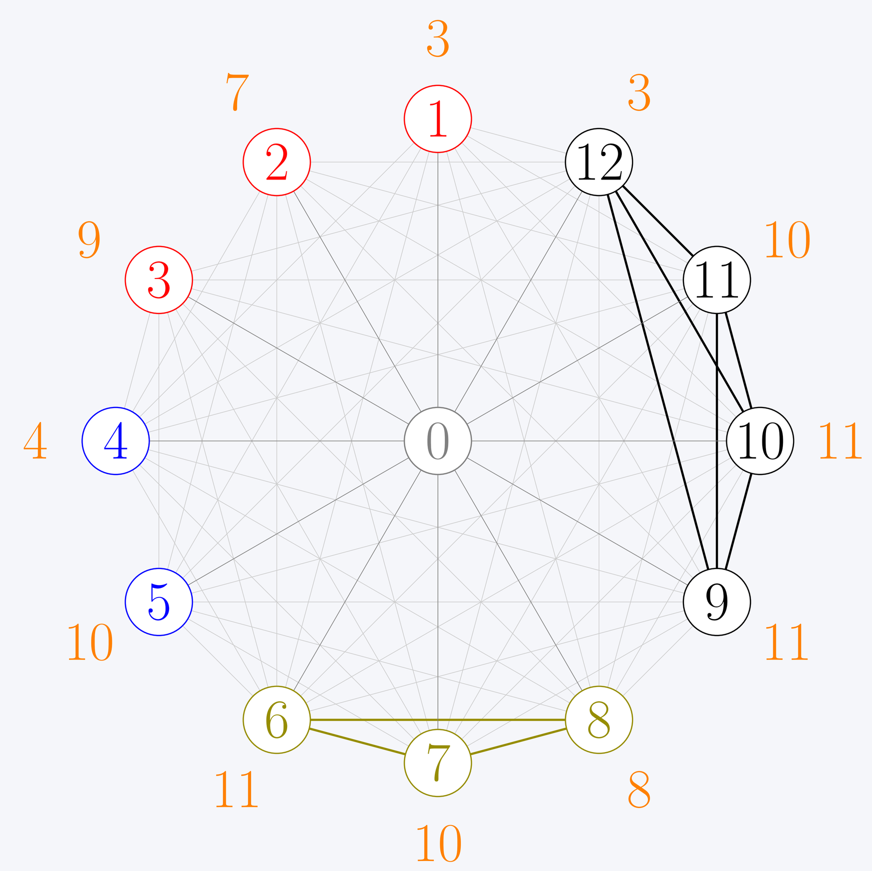
Classical Statistic: level

$\mathsf{level}(\kappa)$ = total # grains - #{non-sink edges}.

Previously known: level corresponds to area when $\mu=(n)$ and $\nu=\emptyset$.

$\kappa(v_i)=j$ + correction term $\iff$ j appears in ith column from the right.





New Statistic: delay

$\mathsf{delay}(\kappa)$ is computed via the following algorithm:

1. Topple 0
2. As long as some vertex hasn't been toppled, traverse the vertices of G in decreasing order.
3. When you arrive at an un-toppled vertex v:
   - if v can topple, topple it and continue.
   - If v *cannot* be toppled, increment the delay by 1 and continue.

Computing delay

The configuration in our example has toppling word $10,9,7,6,5,3,2,11,8,4,1,12$, and hence a **delay** of 6. Following our algorithm step-by-step gives the following:

0. Fire 0.
1. In Round 1, we skip 12, 11, 8, 4, and 1.
2. In Round 2, we skip 12 again, but 11, 8, 4, and 1 topple.
3. In Round 3, 12 topples.

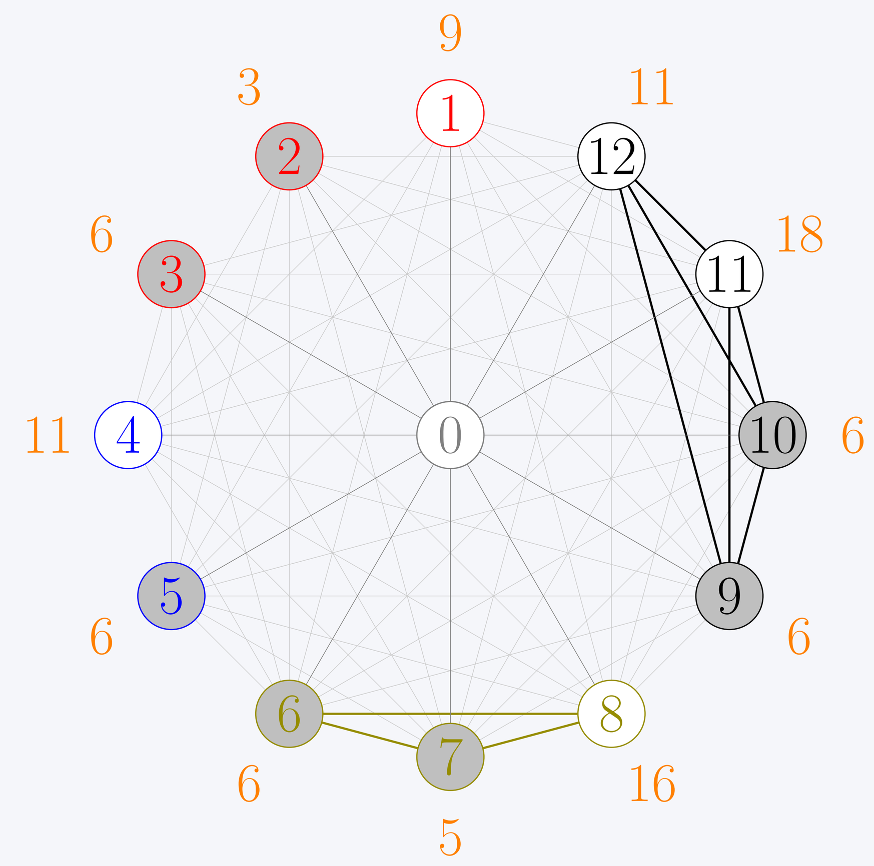


Figure 1. State of the configuration after Round 1. Nodes filled in gray have been toppled and are ignored in subsequent rounds.

Main Result

The bistatistic (**level**, **delay**) allows us to compute coeffs of ∇e_n in terms of graphs.

Theorem (DADILLBVW):
For all n and μ, ν compositions with $|\mu|+|\nu|=n$, we have

$$\langle \nabla e_n, e_\mu h_\nu\rangle = \sum_{\kappa\in \mathsf{sortrec}(\mu;\nu)} q^{\mathsf{level}(\kappa)} t^{\mathsf{delay}(\kappa)},$$

where $\mathsf{sortrec}(\mu;\nu)$ is the set of sorted recurrent configs on $\hat{G}_{\mu,\nu}$.

[1] Erik Carlsson and Anton Mellit, *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31** (2018), no. 3, 661–697. MR 3787405

[2] Nicholas A. Loehr and Jeffrey B. Remmel, *Conjectured combinatorial models for the Hilbert series of generalized diagonal harmonics modules*, Electron. J. Combin. **11** (2004), no. 1, Research Paper 68, 64. MR 2097334