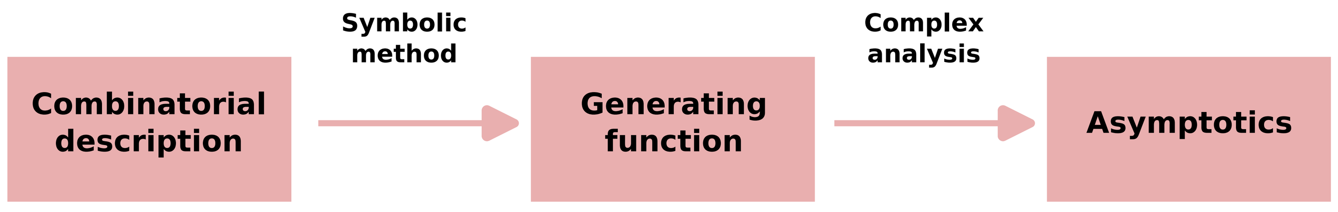
Asymptotics of bivariate algebraico-logarithmic generating functions

Torin Greenwood and Tristan Larson

North Dakota State University

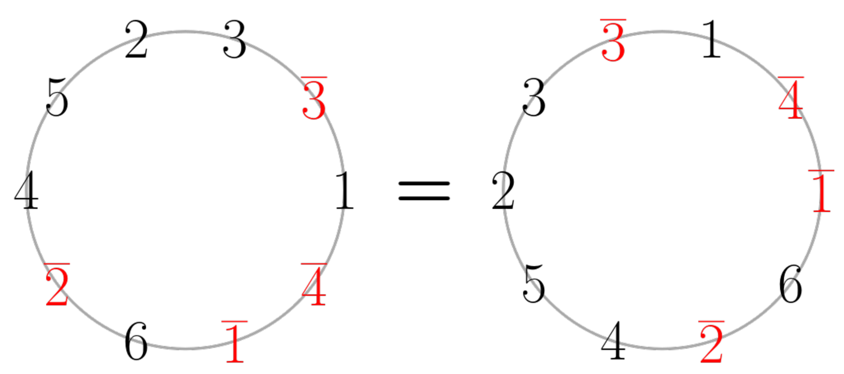
Automating counting

A goal in analytic combinatorics is to find asymptotics for arrays of numbers.



Big Question: Given a combinatorial description of an array, can we automate finding its asymptotics?

Example:



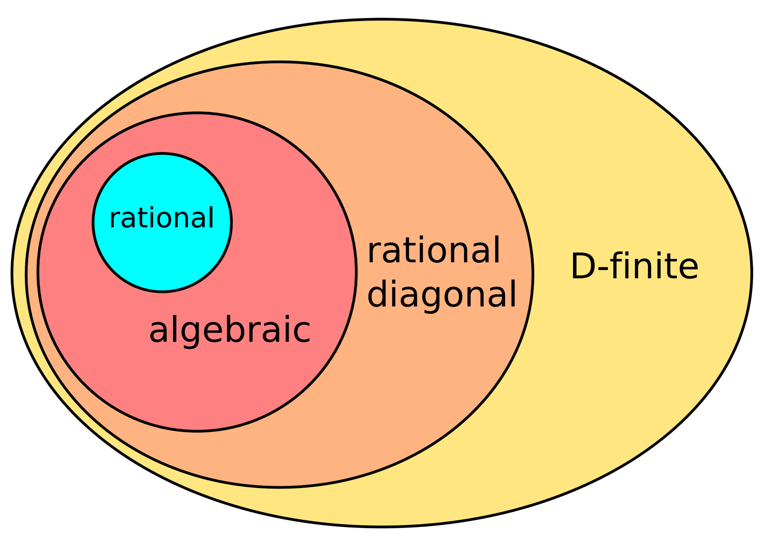
Symbolic method: Circular arrangements of the sets $\{1,\ldots,n\}$ and $\{\bar{1},\ldots,\bar{m}\}$ are cycles of sequences of colored numbers, so the symbolic method enodes in a GF:

$$C(x,y) = \log\left(\frac{1}{1-(x+y)}\right).$$

Asymptotics: Once we have the GF, our result yields as $r\to\infty$

$$[x^r y^{\ell r}]C(x,y) \sim \frac{r^{-3/2}\ell^{-r\ell}(1+\ell)^{(1+\ell)r}}{\sqrt{2\pi\ell(1+\ell)}}.$$

Hierarchy of GFs



- Rational GFs encode the output of deterministic finite automata.
- Algebraic GFs encode outputs of context-free grammars. Examples include Dyck paths, binary trees, constrained (random) walks, and RNA secondary structures.
- D-finite GFs satisfy a linear differential equation and encode sequences which satisfy a polynomial recurrence. Examples include cycles of objects.

Within the class of D-finite GFs are GFs that involve a logarithmic factor. Cases where logarithms may appear include Póyla enumeration, objects involving cycles, or implicitly as components of larger structures.

Background

- Flajolet and Sedgewick's book, [1]: univariate generating functions.
- Pemantle, Wilson, and Melczer's book, [4]: multivariate rational generating functions $F(\mathbf{x}) = \sum_{n_1,\ldots,n_d} a_{i_1,\ldots,i_d} x_1^{n_1}\cdots x_d^{n_d}$. Here, $\mathbf{x} = (x_1,\ldots,x_d)$.
- Analytic combinatorics mantra:
  - Location of a GF's singularities determines exponential growth of its coefficients.
  - Behavior of the GF near its singularities determines subexponential growth.
- The Cauchy integral formula is central to these derivations:

$$\left[x_1^{n_1}\cdots x_d^{n_d}\right]F(\mathbf{x}) = \left(\frac{1}{2\pi i}\right)^d \int_T \frac{F(\mathbf{x})}{x_1^{n_1+1}\cdots x_d^{n_d+1}}\,dx_1\cdots dx_d$$

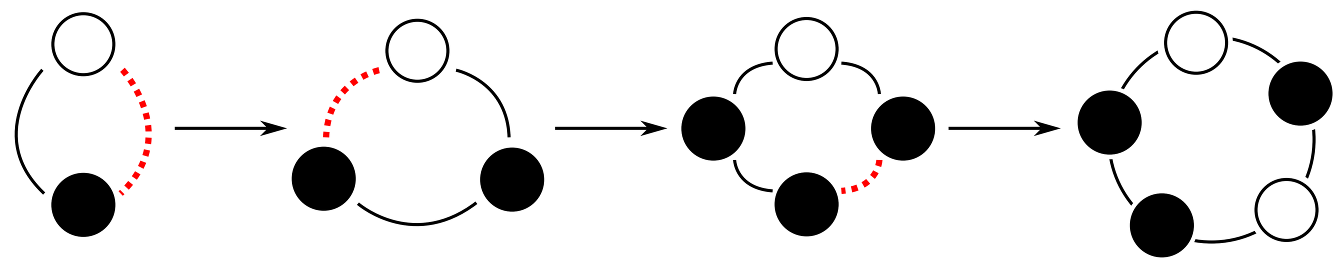
Result

Let $H(x,y)$ be an analytic function near the origin whose power series expansion at $(0,0)$ has non-negative coefficients. Define $\mathcal{V} = \{(x,y) : H(x,y) = 0\}$. Assume that there is a single smooth strictly minimal critical point of $\mathcal{V}$ at (p,q) within the domain of analyticity of H where p and q are real and positive. Let $\lambda = \frac{r+O(1)}{s}$ as $r,s\to\infty$ with r and s integers.

Assume that $H_x(p,q)$ is nonzero. Fix $\alpha\in\mathbb{R}$ where $\alpha\notin\mathbb{Z}_{\le 0}$ and $\beta\in\mathbb{Z}_{\ge 0}$. Then, for certain constants $M\neq 0$ and $\mathcal{E}_j$, the following expression holds as $r,s\to\infty$:

$$[x^r y^s]H(x,y)^{-\alpha}\log^{\beta}(H(x,y)) \sim (-1)^{\beta}\frac{(-pH_x(p,q))^{-\alpha}r^{\alpha-1}}{\Gamma(\alpha)\sqrt{-2\pi q^2 M r}}p^{-r}q^{-s}\log^{\beta} r\left[1+\sum_{j\ge 1}\frac{\mathcal{E}_j}{\log^j r}\right],$$

Example: Necklaces



The necklace process, described in [3], constructs necklaces with black and white beads where no two white beads are adjacent. We consider the GF

$$N(x,y) = \sum_{k\ge 1}\frac{\varphi(k)}{k}\log\left(\frac{1-x^k}{1-x^k-y^k x^{2k}}\right),$$

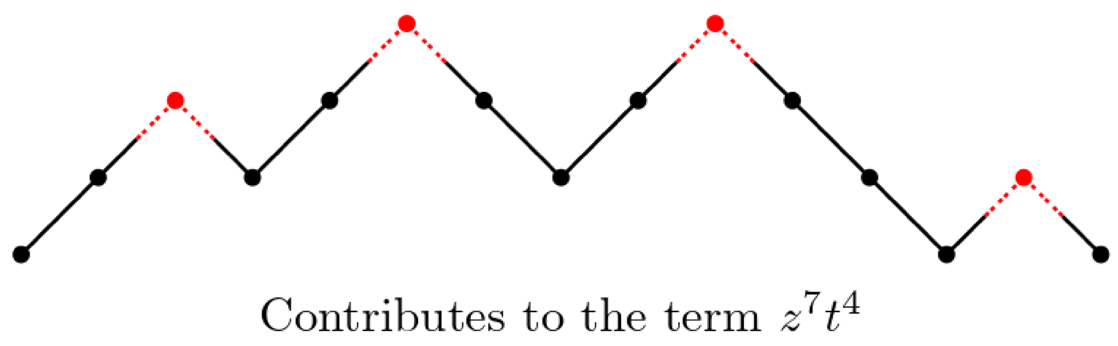
in which the coefficient $[x^r y^s]N(x,y)$ counts the number of necklaces with r total beads and s white beads. To determine growth of coefficients in the direction $(\ell,1)$, with $\ell>2$:

1. **Identify smooth critical points** (SCPs) by solving the systems $[H_k = 0, x\frac{\partial}{\partial x}H_k - \ell y\frac{\partial}{\partial y}H_k]$, where here $H_k = 1-x^k-y^k x^{2k}$.
2. **Rule out other types of critical points** (via the implicit function theorem) by showing $[H_k = 0, \frac{\partial}{\partial x}H_k = 0, \frac{\partial}{\partial y}H_k = 0]$ has no solutions.
3. **Determine minimal SCPs** meaning no singularities have coordinate-wise smaller moduli than these critical points. This can be the most computationally intensive part but is simpler if the GF is combinatorial (non-negative coefficients).
4. **Apply the asymptotic result**.

This process yields

$$[x^{\ell n}y^n]N(x,y) \sim \frac{n^{-3/2}\ell^{5/2}}{\sqrt{2\pi}}\frac{(\ell-1)^{(2\ell n-2n+3)/2}}{(\ell-2)^{(2\ell n-4n+9)/2}}.$$

Example: Log of Narayana Numbers



Here, an algebraic singularity determines asymptotics for a non-algebraic GF. The Narayana numbers count Dyck paths with n steps and k peaks, encoded by

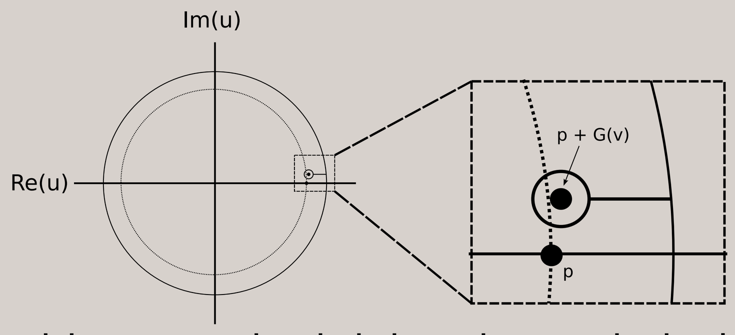
$$N(z,t) = \frac{1+z-tz-\sqrt{(1+z-tz)^2-4z}}{2z}.$$

We consider the growth rate of coefficients $[z^n t^s]\log^r N(z,t)$ in the direction $(\ell,1)$ for $\ell>1$. The dominant asymptotics are determined by the algebraic singularity of $N(z,t)$ at the point $(p,q) := ([1-1/\ell]^2, 1/[\ell-1]^2)$, and we find no nonsmooth critical points. We then expand $\log^r N(z,t)$ at (p,q) so that Corollary 2 of [2] can be applied. This yields our final result:

$$[z^{\ell n}t^n]\log^r N(z,t) \sim \frac{r}{2\pi}\log^{r-1}\left(\frac{\ell}{\ell-1}\right)\cdot n^{-2}\cdot(\ell-1)^{-2n(\ell-1)-1}\ell^{2\ell n-1}.$$

Proof Outline

Step 1: Change of variables
Step 2: Choose a convenient contour
Step 3: Approximate the integrand with a product integral
Step 4: Evaluate the product integral



- Compared to previous results, adding logarithms required tightening technical error bounds and deriving several new approximations of logarithmic factors.
- When $\beta\neq 0$, we obtain an *asymptotic series*.

arXiv + References

Find our work on arXiv:
`https://arxiv.org/abs/2405.08133`

[1] Phillipe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, 2009.

[2] Torin Greenwood. Asymptotics of bivariate analytic functions with algebraic singularities. *Journal of Combinatorial Theory, Series A*, 153:1–30, 2018.

[3] Benjamin Hackl and Helmut Prodinger. The necklace process: a generating function approach. *Statist. Probab. Lett.*, 142:57–61, 2018.

[4] Robin Pemantle, Mark C. Wilson, and Stephen Melczer. *Analytic Combinatorics in Several Variables*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2024.

torin.greenwood@ndsu.edu
tristan.larson@ndsu.edu

FPSAC 2024

NDSU NORTH DAKOTA STATE UNIVERSITY