Slit-slide-sew bijections for constellations and quasiconstellations

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Constellations

p-hypermap (for $p \geq 2$): plane map with dark faces (a.k.a *hyperedges*) and light faces (a.k.a *hyperfaces*) such that:

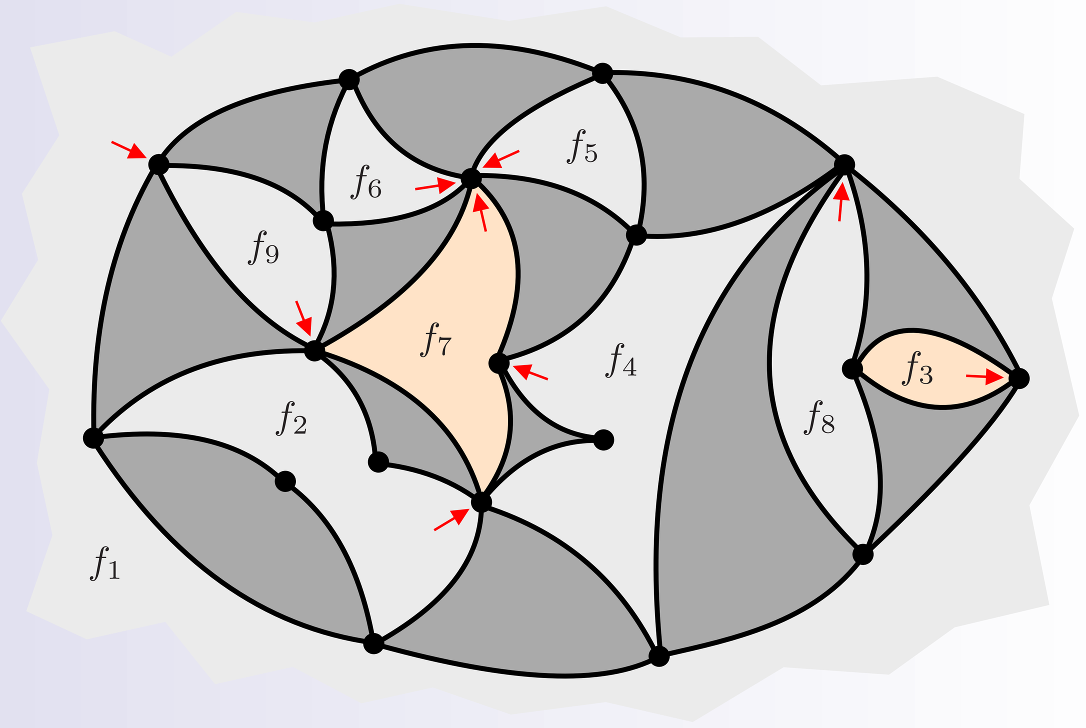
- adjacent faces have different shades;
- dark faces have degree p.

p-hypermap of type $\boldsymbol{a} = (a_1, \dots, a_r)$: light faces $f_1, \dots, f_r$ of respective degrees $a_1, \dots, a_r$.

Flawed face: light face whose degree $\notin p\mathbb{Z}$.

p-constellation: no flawed faces.

Quasi-p-constellation: exactly two flawed faces.



Enumeration

For $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, define the following:

- $C(\boldsymbol{a})$: number of p-*hypermaps of type* $\boldsymbol{a}$.
- $E(\boldsymbol{a}) := \sum_{i=1}^{r} a_i$: number of *edges*.
- $D(\boldsymbol{a}) := \frac{E(\boldsymbol{a})}{p}$: number of *dark faces*.
- $V(\boldsymbol{a}) := E(\boldsymbol{a}) - D(\boldsymbol{a}) - r + 2$: number of *vertices*.

Thm ([Bousquet-Mélou–Schaeffer '00, Collet–Fusy '14]). *For p-constellations or quasi-p-constellations*

$$C(\boldsymbol{a}) = c_{\boldsymbol{a}} \cdot \frac{(E(\boldsymbol{a}) - D(\boldsymbol{a}) - 1)!}{V(\boldsymbol{a})!} \prod_{i=1}^{r} \alpha(a_i),$$

$$\text{where} \quad \alpha(x) := \frac{x!}{\lfloor x/p \rfloor! \, (x - \lfloor x/p \rfloor - 1)!},$$

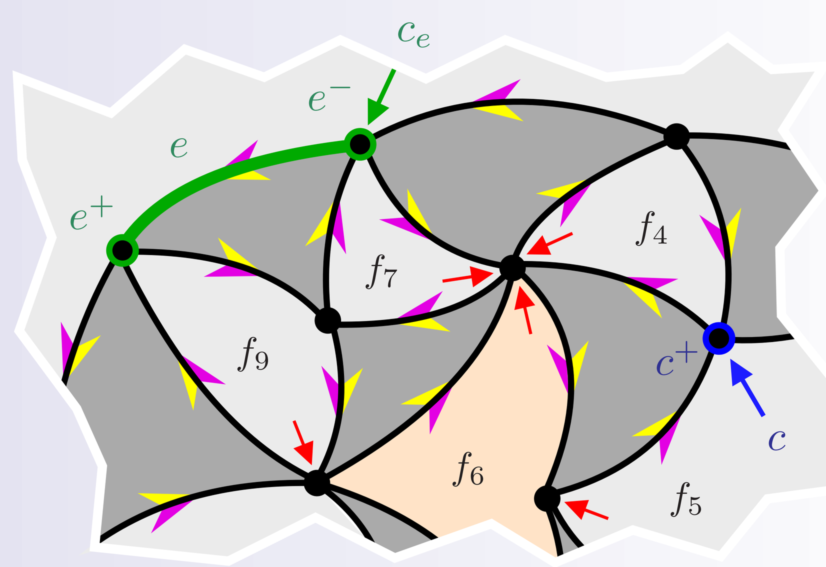
and $c_{\boldsymbol{a}}$ equals 1 for p-constellations, or $p-1$ for quasi-p-constellations.

Orientation

Edge orientation. Two possibilities:

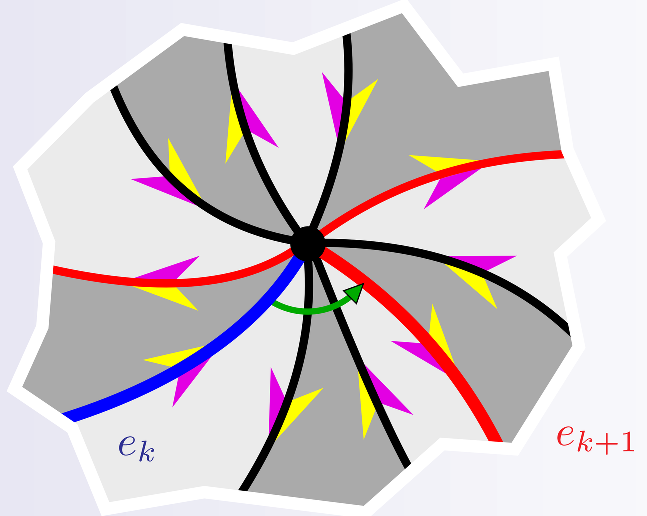
- light-left: light faces *on the left* of edges;
- light-right: light faces *on the right* of edges.

This defines an **oriented metric** $\vec{\mathrm{d}}$.

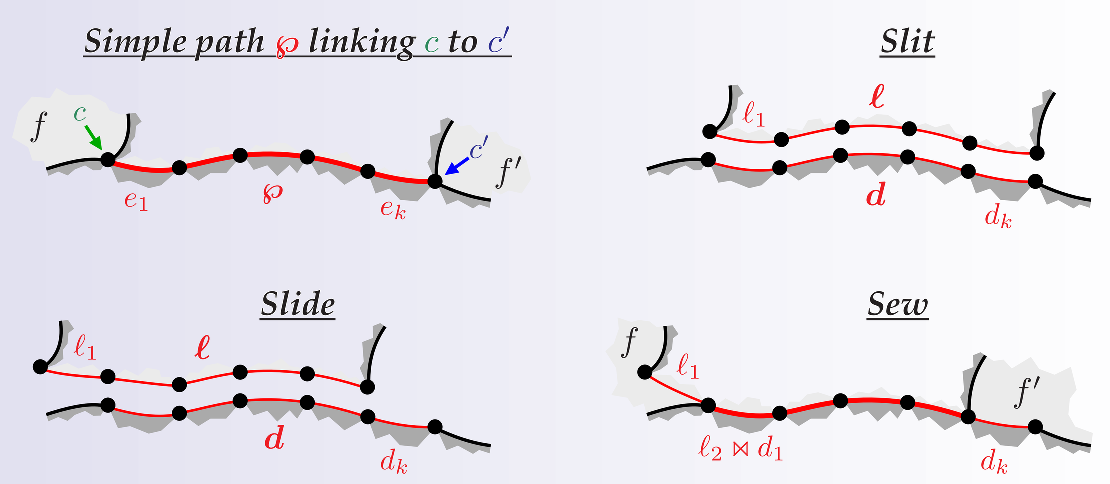


Geodesic: shortest **oriented** path linking two given vertices.

Lightest geodesic: leftmost (resp. rightmost) geodesic if the considered orientation is the light-left (resp. light-right).



Methodology: *slit slide sew*

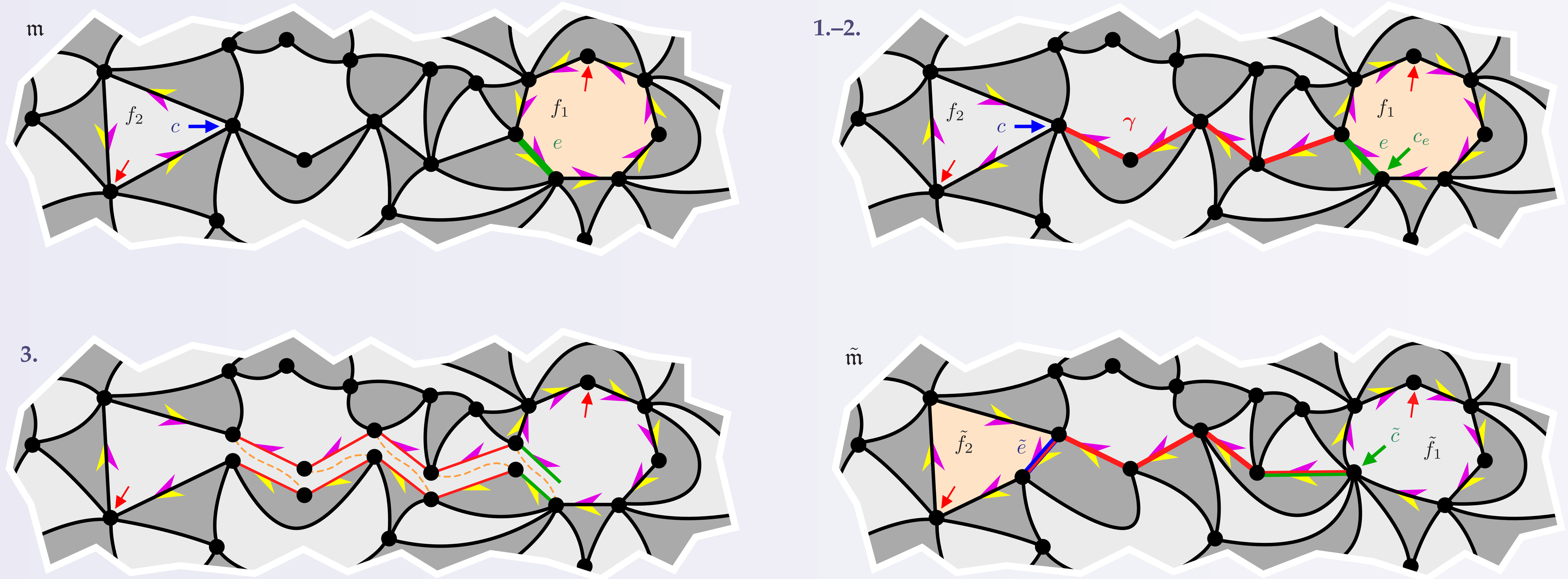


Bijective interpretation

$$\boldsymbol{a} = (a_1, \dots, a_r) \equiv (k, -k, 0, \dots, 0) \mod p, \quad \text{with } a_1 \geq 2$$

$$\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_r) := (a_1 - 1, a_2 + 1, a_3, \dots, a_r)$$

$$\underbrace{(a_1 - \lceil a_1/p \rceil)}_{\substack{\text{edge in } f_1 \\ \text{leaving } c}} \underbrace{(a_2 + 1)}_{\substack{\text{corner} \\ c \text{ in } f_2}} C(\boldsymbol{a}) = \underbrace{(\tilde{a}_1 + 1)}_{\substack{\text{corner} \\ \tilde{c} \text{ in } f_1}} \underbrace{(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil)}_{\substack{\text{edge in } \tilde{f}_2 \\ \text{leaving } \tilde{c}}} C(\tilde{\boldsymbol{a}})$$

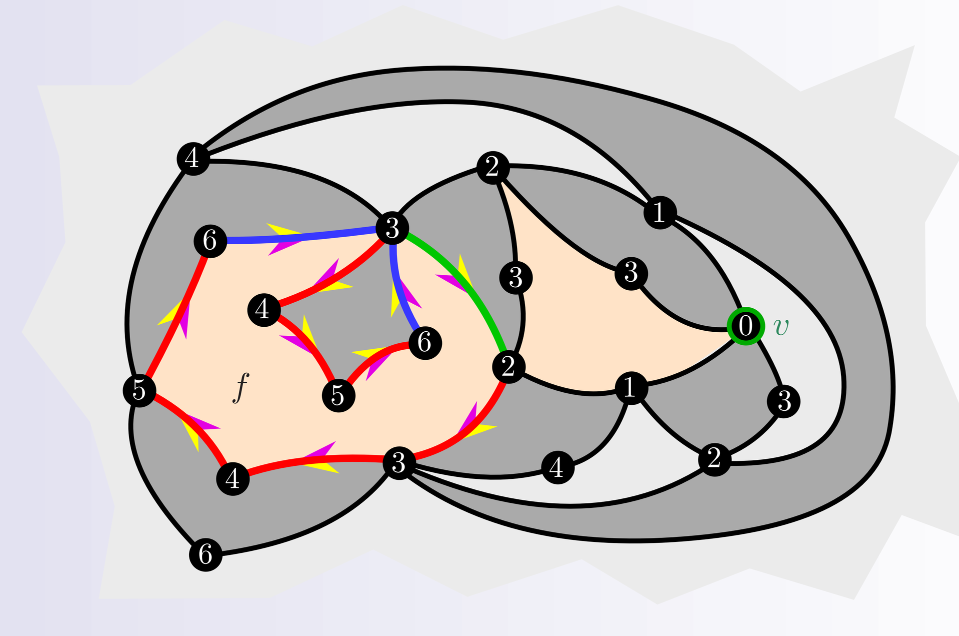


1. Reverse the orientation.
2. Consider the corner c_e preceding e and the lightest geodesic γ from c_e to c^+.
3. Slit, slide, sew along γ from c_e to c: the light side of an edge is now matched with the dark side of the previous edge.
4. In the resulting map $\tilde{\mathfrak{m}}$, denote by $\tilde{c}$ the corner corresponding to the unmatched light side of the first edge of γ, as well as $\tilde{e}$ the edge corresponding to the unmatched dark side of the final edge of γ.

Edge types

Edge types. Three types of edges: the edge e is

- leaving v if $\vec{\mathrm{d}}(v, e^+) = \vec{\mathrm{d}}(v, e^-) + 1$;
- approaching v if $\vec{\mathrm{d}}(v, e^+) = \vec{\mathrm{d}}(v, e^-) + 1 - p$;
- irregular w.r.t. v if $\vec{\mathrm{d}}(v, e^+) - \vec{\mathrm{d}}(v, e^-) \not\equiv 1 \mod p$.



Prop. *In a p-constellation or quasi-p-constellation, consider a vertex v and a light face f of degree a, flawed in the case of a quasi-p-constellation. Then, among the a edges incident to f:*

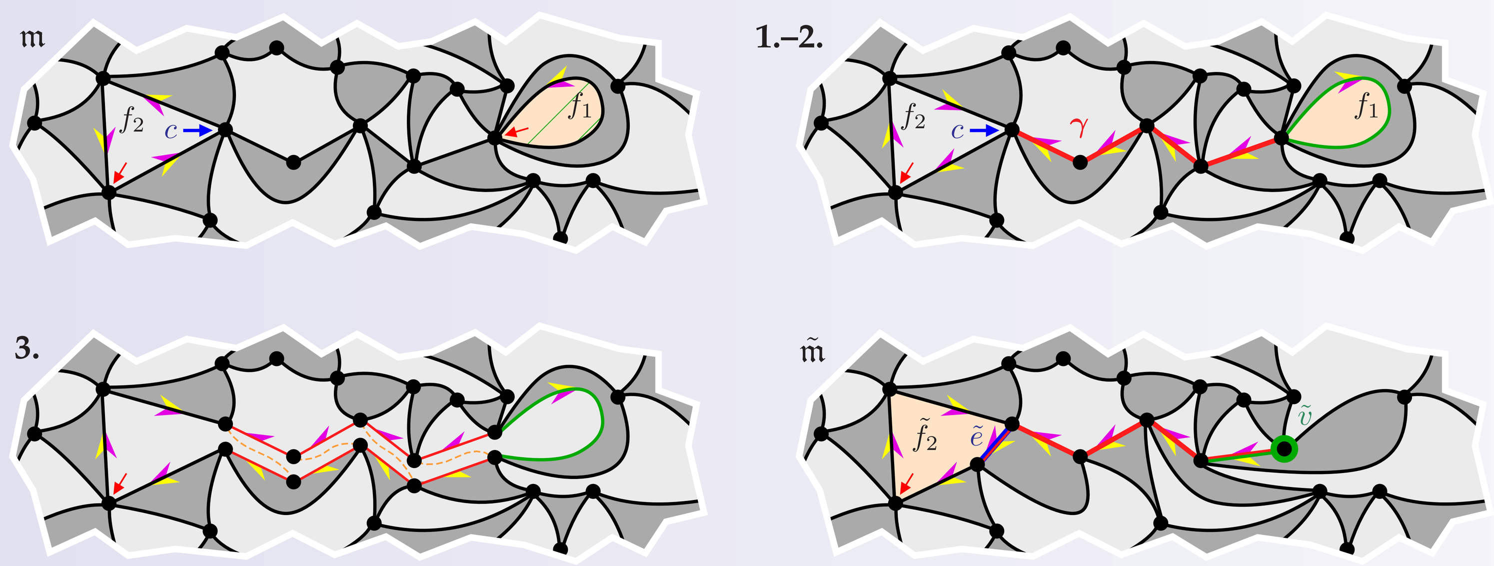
- *$(a - \lceil a/p \rceil)$ are leaving v;*
- *$\lfloor a/p \rfloor$ are approaching v.*

Case of a face of degree one

$$\boldsymbol{a} = (1, a_2, \dots, a_r) \equiv (1, -1, 0, \dots, 0) \mod p,$$

$$\tilde{\boldsymbol{a}} = (\tilde{a}_2, \dots, \tilde{a}_r) := (a_2 + 1, a_3, \dots, a_r)$$

$$\underbrace{(a_2 + 1)}_{\substack{\text{corner} \\ c \text{ in } f_2}} C(\boldsymbol{a}) = \underbrace{(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil)}_{\substack{\text{edge in } \tilde{f}_2 \\ \text{leaving } v}} \underbrace{V(\tilde{\boldsymbol{a}})}_{\substack{\text{distinguished} \\ \text{vertex } v}} C(\tilde{\boldsymbol{a}})$$



Suppressing f_1.

1. Reverse the orientation.

2.–3. Slit, slide, sew along the lightest geodesic γ from the unique corner of f_1 to c^+.

4. In the resulting map $\tilde{\mathfrak{m}}$, denote by $\tilde{v}$ the origin of γ, and $\tilde{e}$ the final edge of γ.

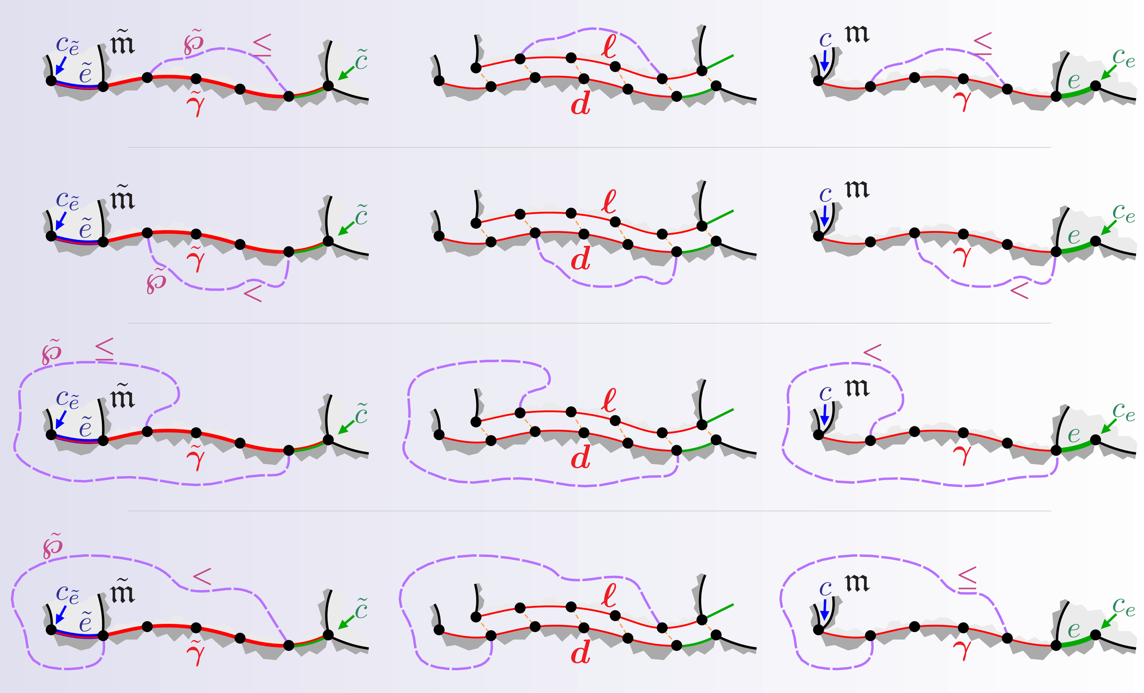
Adding a face.

1. Reverse the orientation.

2.–3. Slit, slide, sew along the lightest geodesic from the corner preceding $\tilde{e}$ to $\tilde{v}$ **without disconnecting the map** at $\tilde{v}$.

4. Denote by f_1 the face that appears at the tip of the sliding path and by c the corner at its origin.

Conservation of sliding paths



- If the sliding path $\tilde{\wp} \neq \tilde{\gamma}$, it uses a purple dashed circumventions.
- Tracking it back to the original map $\mathfrak{m}$ yields a circumvention providing a better option than γ.

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