The commutant of divided difference operators, Klyachko's genus, and the comaj statistic

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Goals

- Identify *all* operators on Schubert symbols that commute with the "partial" operators $\{\partial_w\}$, generalizing [HPSW]
- Determine which of those give differentials ($d(pq) = dp\, q + p\, dq$)
- Find relationships among known **genera** (ring maps from cohomology)
- Exploit genera to compute/constrain Schubert structure constants

Dramatis Personæ

D	Dynkin diagram
$A_{\mathbb{Z}^+}$	type A Dynkin diagram with $\mathbb{Z}^+$ as nodes
$A_{\mathbb{Z}}$	type A Dynkin diagram with $\mathbb{Z}$ as nodes
$W(D)$	associated Weyl group
$H(D)$	cohomology ring of flag variety associated with D
r_α or r_i	simple reflection associated to simple root α or node i
$RW(\pi)$	set of reduced words for $\pi \in W(D)$
$H(A_{\mathbb{Z}^+})$	generated by $\{S_{r_i} : i \in \mathbb{Z}^+\}$, ($r_i$ simple reflection)
$H(A_{\mathbb{Z}})$	generated by $\{S_{r_1 r_2 \cdots r_k}\}$

Definition (Genera of the cohomology ring)

We call a ring homomorphism from $H(D)$ to another ring a **genus**.

Example (The Lascoux-Schützenberger genus)

Define $H(A_{\mathbb{Z}^+}) \longrightarrow \mathbb{Z}[x_1, x_2, \dots]$ by

$$S_\pi \mapsto S_\pi(x_1, x_2, \dots),$$

sending each Schubert symbol to the corresponding "Schubert polynomial" in variables $x_1, x_2, \dots$. This map is an isomorphism.

Example (The Klyachko genus)

The **Klyachko genus** is the map

$$H(A_{\mathbb{Z}}) \xrightarrow{\iota^*} K := \mathbb{Q}[\dots, k_{-1}, k_0, \dots] \Big/ \left\langle k_i \left(k_i - \frac{k_{i-1} + k_{i+1}}{2}\right), \forall i \in \mathbb{Z} \right\rangle$$

$$S_\pi \mapsto \frac{1}{\ell(\pi)!} \sum_{Q \in RW(\pi)} \prod_{q \in Q} k_q$$

Klyachko considered the inclusion ι of the permutahedral toric variety (AKA the regular semisimple Hessenberg variety) into the flag manifold. The above map is related to a biinfinite limit of the induced map ι^* on H^*.

Example (The affine-linear genus)

Define $\gamma : H(A_{\mathbb{Z}}) \longrightarrow \mathbb{Q}[a, b]$ by

$$\mathcal{S}_\pi \longmapsto \frac{1}{\ell(\pi)!} \sum_{P \in RW(\pi)} \prod_{i \in P} (ai + b);$$

with $RW(\pi)$ the set of reduced words for π.

Applying the $a = 0, b = 1$ case to $S_\pi S_\rho$ leads to Nenashev's theorem:

Theorem (First Rectification Theorem (Nenashev))

Let $RW(\pi)$ denote the set of reduced words for π. Then there exists (although the proof **doesn't find one***) some "rectification" map*

$$\{\textit{shuffles of any word in } RW(\pi) \textit{ with any word in } RW(\rho)\} \to \coprod_\sigma RW(\sigma)$$

whose fiber over any reduced word for σ has size $c^\sigma_{\pi\rho}$, the coefficient from $\mathcal{S}_\pi \mathcal{S}_\rho = \sum_\sigma c^\sigma_{\pi\rho} \mathcal{S}_\sigma$.

Partials, martials, and nil Hecke actions

$Nil(D)$ (nil Hecke algebra) is the formal linear combos of $\{d_\pi : \pi \in W(D)\}$ with product

$$d_\pi d_\rho := \begin{cases} d_{\pi\rho} & \text{if } \ell(\pi\rho) = \ell(\pi) + \ell(\rho) \\ 0 & \text{if } \ell(\pi\rho) < \ell(\pi) + \ell(\rho). \end{cases}$$

For α a vertex of D, define the "partial" operator $\partial_\alpha \circlearrowright H(D)$ by

$$\partial_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{\pi r_\alpha} & \text{if } \pi r_\alpha < \pi \\ 0 & \text{if } \pi r_\alpha > \pi \end{cases}$$

We introduce the **martial** operator $\reflectbox{$\partial$}_\alpha$ by

$$\reflectbox{∂}_\alpha \mathcal{S}_\pi := \begin{cases} \mathcal{S}_{r_\alpha \pi} & \text{if } r_\alpha \pi < \pi \\ 0 & \text{if } r_\alpha \pi > \pi \end{cases}$$

For $w = \prod Q$ with Q a reduced word, the operators $\partial_w := \prod_{q \in Q} \partial_q$ and $\reflectbox{$\partial$}_{w^{-1}} := \prod_{q \in Q} \reflectbox{∂}_q$ are independent of Q.

- The algebra $Nil(D)$ acts on $H(D)$ via $d_\pi \mapsto \reflectbox{∂}_\pi$.
- The opposite algebra $Nil(D)^{opp}$ acts on $H(D)$ via $d_\pi \mapsto \partial_\pi$.

Theorem (Commuting Actions)

The action of $Nil(D)$ on $H(D)$ commutes with the $Nil(D)^{op}$-action. Furthermore, each operator on $H(D)$ that commutes with all operators ∂_α arises as the action of a unique element of $Nil(D)$.

One says that $Nil(D)$ and $Nil(D)^{op}$ are each other's **commutants** *on $H(D)$.*

Theorem (The differentials that commute with $\{\partial_i\}$)

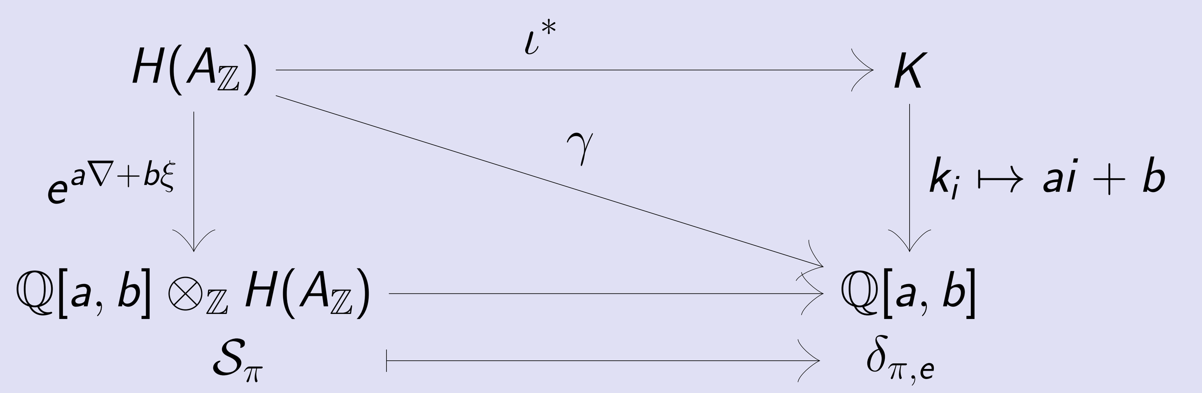
Let $\sum c_\alpha \reflectbox{∂}_\alpha \in Nil(D)$ be a degree -1 Leibniz differential. If D is simply laced, then $c_\alpha = \frac{1}{2}\sum_\beta c_\beta$ with sum over neighboring nodes. Thus for D of finite type, $c_\alpha = 0$ for all α. For $D = A_{\mathbb{Z}^+}$, $c_i \equiv i$ (or a multiple) and for $D = A_{\mathbb{Z}}$, either $c_i \equiv i$ or $c_i \equiv 1$ (or a linear combination of the two). Thus the only differentials of degree -1 are linear combinations of

$$\nabla = \sum_m m \reflectbox{∂}_m \quad (D = A_{\mathbb{Z}} \text{ or } A_{\mathbb{Z}^+}) \qquad \xi = \sum_m \reflectbox{∂}_m, \quad (D = A_{\mathbb{Z}})$$

∇ was found in [HPSW]. ξ was found in [N] and characterized differently. One can exponentiate differentials to get ring automorphisms.

Theorem (Relations between the genera)

The following triangles commute:



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Definition (barred words, comaj, and q-statistic)

comaj(P)	sum of positions of ascents in reduced word P, e.g. comaj(6456) $= 2 + 3$
barred word P for π	reduced word in which some letters are overlined, e.g. $12\overline{1}$ for (13)
q-statistic of barred word	(sum of barred letters) + comaj; e.g. q-statistic of $31\overline{4}$ is $4 + 2$
shuffle $\sqcup\!\sqcup$ of a pair (P, R)	permutation of $P \coprod R$, preserving order of each of P and R
inversions in shuffle $\sqcup\!\sqcup$	for a pair (P, R), the number of "a letter in R leftward of a letter in P"
q-statistic of a shuffle $\sqcup\!\sqcup$	sum of the q-statistics of P and R, plus the # of inversions in $\sqcup\!\sqcup$

Example (q-Klyachko genus (Nadeau-Tewari))

$$\gamma_q : H(A_{\mathbb{Z}}) \to \mathbb{Q}(q)[\alpha, \beta] \qquad \text{defined by} \qquad S_\pi \mapsto \frac{1}{\ell(\pi)^{q}_{\bullet}} \sum_{P \in RW(\pi)} q^{\mathrm{comaj}(P)} \prod_{i \in P} (\alpha q^i + \beta)$$

where $k^{q}_{\bullet} := 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1})$.

We apply the Nadeau-Tewari genus to $S_\pi S_\rho$ to obtain a more refined rectification theorem:

Theorem (Second Rectification Theorem)

There **exists** *(but the proof doesn't find one) a "rectification" map*

$$\{\textit{shuffles of pairs } (P, R) \textit{ of barred words for } \pi, \rho\} \quad \to \quad \coprod_\sigma \{\textit{barred words for } \sigma\}$$

preserving # of bars and q-statistic, *whose fiber over each reduced word for σ has size $c^\sigma_{\pi\rho}$, the coefficient from $S_\pi S_\rho = \sum_\sigma c^\sigma_{\pi\rho} S_\sigma$.*

Example (Second Rectification Theorem)

Let $\pi = \rho = [12463578] \in S_8$ (length 3). We find the fiber over each fully barred σ.

Each of $\pi = \rho$ has two (fully-barred) reduced words, $\overline{354}$ and $\overline{534}$, with comaj$(\overline{354}) = 1$ and comaj$(\overline{534}) = 2$. The sum of the barred numbers is 12 for both. For each (P, R), there are $\binom{6}{3}$ ways to shuffle. Thus: $2 \cdot 2 \cdot \binom{6}{3} = 80$ shuffles $\sqcup\!\sqcup$; the q-statistics range from

$$26 = 1 + 12 + 1 + 12 + 0 \qquad \text{for } (\overline{354}, \overline{354}) \text{ with trivial shuffle } PR \text{, to}$$

$$37 = 2 + 12 + 2 + 12 + 3 \cdot 3 \qquad \text{for } (\overline{534}, \overline{534}) \text{ for reverse shuffle } RP.$$

There are 7 terms $\mathcal{S}_\sigma$ in $\mathcal{S}_\pi \mathcal{S}_\rho$ (one with coefficient 2) with various numbers of reduced words.

q-statistic:	26	27	28	29	30	31	32	33	34	35	36	37	
	1	3	5	8	11	12	12	11	8	5	3	1	total = 80
σ													
[23561478]	1	1	2	1	2	1	1						
[14562378]		1		1	1	1		1					
[13572468]			1	2	2	3	3	2	2	1			
[13572468]			1	2	2	3	3	2	2	1			again
[23471568]		1	1	2	2	2	1	1					
[13482567]					1	1	2	2	2	1	1		
[12673458]					1		1	1	1		1		
[12583467]						1	1	2	1	2	1	1	

In line 1, we list the # of fully barred shuffles with given q-statistic (totalling 80). Each subsequent row lists σ and the # of fully barred words for it with given q-statistic. The second rectification theorem asserts that the # atop each column is the sum of those below.

References

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