Skein Relations for Punctured Surfaces

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Cluster Algebras from Surfaces

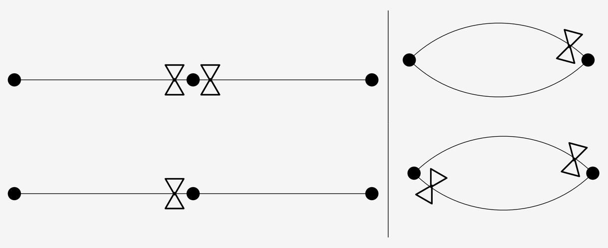
Cluster algebras are recursively defined commutative rings with a set of distinguished generators called cluster variables, which appear in fixed-size subsets known as clusters [3]. Triangulated surfaces provide a geometric model for ordinary cluster algebras of surface type, where the clusters correspond to triangulations and the cluster variables correspond to arcs between these marked points [1].

Tagged Triangulations

To create complete geometric models for cluster algebras from punctured surfaces, [2] introduced the concept of tagged arcs. A tagged arc is an arc with ends tagged as either plain or notched and must not form a once-punctured monogon. We write $\eta^{(p)}$ to indicate when η has a single notched end at p and $\eta^{(pq)}$ when it is notched at both ends, p and q.

Compatible Examples:

Incompatible Examples:



Poset Construction

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of the surface (S, M). For any arc γ on (S, M), we construct a corresponding poset P_γ as described in [6, 7]. These posets P_γ correspond exactly to the posets of join-irreducibles in the lattice of perfect matchings of the snake graph G_γ, as outlined in [5, 8].

Example

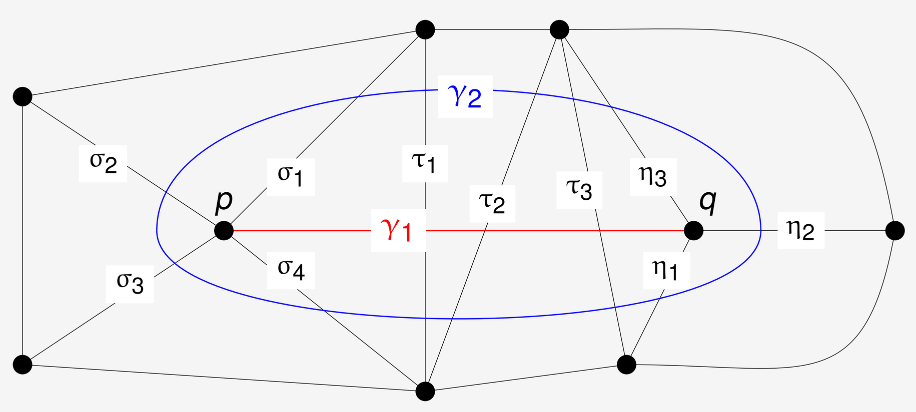
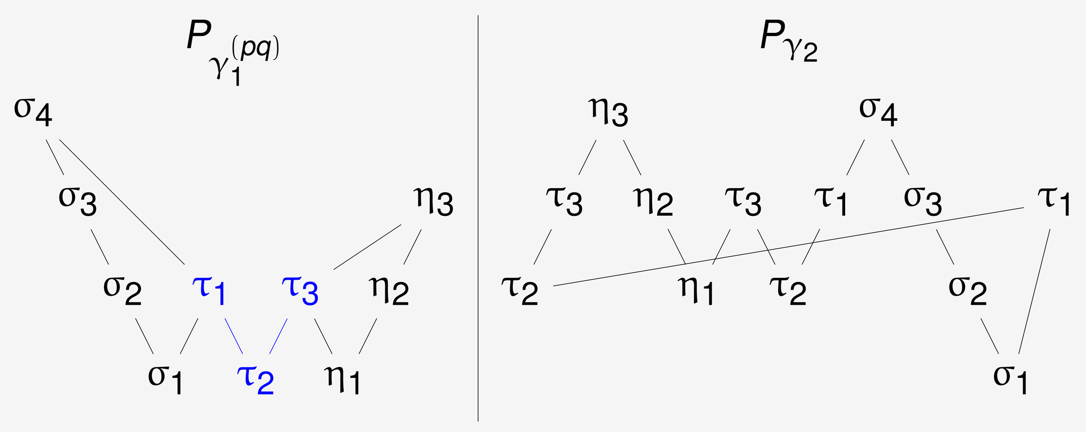


Figure: An example of an arc γ_1 and closed curve γ_2 on a triangulated surface.



The loop fence poset $P_{\gamma_1^{(pq)}}$ and circular fence poset P_{γ_2} for the arcs from the above figure. Note that the fence poset P_{γ_1} for the plain arc γ_1 appears as a subposet of $P_{\gamma_1^{(pq)}}$, indicated in blue.

Poset Expansion

Let γ be an arc or closed curve on a marked surface (S, M) with a triangulation T such that $\gamma \notin T$. Then, the corresponding element x_γ of the cluster algebra $\mathcal{A}(S, M)$, expressed in terms of the cluster associated with T, can be written as

$$x_\gamma^T = \mathbf{x}^{\mathbf{g}_\gamma} \sum_{I \in J(P_\gamma)} w(I).$$

where $J(P)$ denote the poset of lower order ideals of a poset P_γ, $\mathbf{g}_\gamma$ is discussed below, and $w(I)$ denotes an associated weight of $I \in J(P)$ defined as $w(I) = \prod_{j \in I} \left(x_{CCW(\tau_{i_j})} / x_{CW(\tau_{i_j})} \right) y_{\tau_{i_j}}$.

In the above example, the arc $\gamma_1^{(pq)}$ corresponds to

$$x_{\gamma_1^{(pq)}} = \frac{x_{\tau_1} x_{\tau_3}}{x_{\sigma_1} x_{\tau_2} x_{\eta_1}} \left[1 + \frac{x_{\sigma_4} y_{\sigma_1}}{x_{\tau_1} x_{\sigma_2}} + \frac{y_{\tau_2}}{x_{\tau_1} x_{\tau_3}} + \frac{x_{\eta_3} y_{\eta_1}}{x_{\tau_3} x_{\eta_2}} + \frac{x_{\sigma_4} y_{\sigma_1} y_{\tau_2}}{x_{\tau_1}^2 x_{\tau_3} x_{\sigma_2}} + \frac{x_{\sigma_1} x_{\sigma_4} y_{\sigma_1} y_{\sigma_2}}{x_{\tau_1} x_{\sigma_2} x_{\sigma_3}} + \cdots \right]$$

G-Vector

If γ is notched at one or both endpoints, let γ^0 be γ with a plain tag at both endpoints. The vector $\mathbf{g}_\gamma$ is computed as follows:

- minimal elements in P_{γ^0} contribute negatively;
- maximal elements in P_{γ^0} that cover two elements contribute positively; and
- arcs counterclockwise (clockwise) from a plain (notched) endpoint of γ contribute positively (negatively).

The vector $\mathbf{g}_\gamma$ encodes

1. the weight of the minimal matching of the snake graph $G_{\gamma,T}$,
2. the shear coordinate of γ with respect to T, and
3. a minimal injective presentation of the arc module associated to γ.

Construction

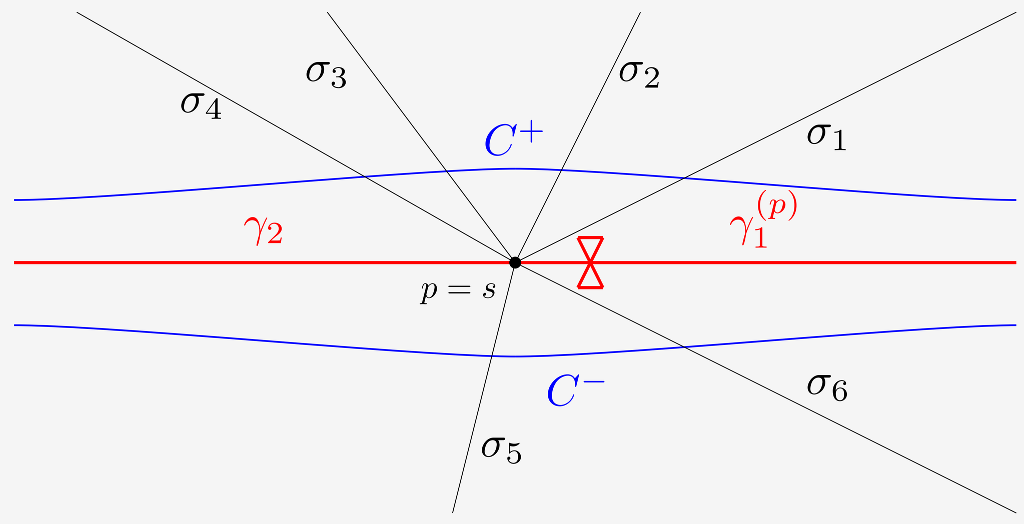
Consider two curves, γ_1 and γ_2, with an incompatibility point s. Sometimes, γ_1 and γ_2 cross the same set of arcs before or after s, and we call this common set of arcs R. We define the sweep set, denoted Sw, as the set of arcs an arc in the resolution pivots past clockwise at a plain endpoint or counterclockwise at a notched endpoint. In this resolution, the sets of arcs (multicurves) are labeled C^+ and C^-, where C^- is the set that does not cross any arcs in R or Sw.

Main Theorem

The fifteen cases described in [4] can be regrouped as transverse crossings of (1) a single arc (2) two arcs and (3) an arc and a closed curve as well as (4) incompatibility at a puncture. The resolution of any such incompatibility yields the following multiplication formula.

1. Let $\{\gamma_1, \gamma_2\}$ be a multicurve of arcs or closed curves which are incompatible. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1} x_{\gamma_2} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.
2. Let γ_1 be an arc or closed curve which is incompatible with itself. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.

Case 1: Incompatibility at Punctures



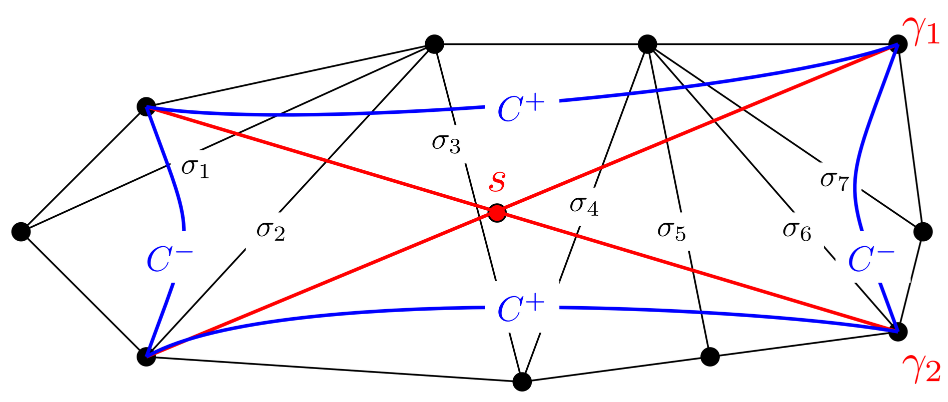
$$R = \emptyset$$

$$Sw = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$$

Here, the resolution will be written as

$$x_{\gamma_1^{(p)}} x_{\gamma_2} = x_{C^+} + y_1 y_2 y_3 y_4 x_{C^-}$$

Case 2: Transverse Crossings



$$R = \{\sigma_3, \sigma_4, \sigma_5\}$$

$$Sw = \{\sigma_6\}$$

The skein relation will be written as

$$x_{\gamma_1} x_{\gamma_2} = x_{C^+} + y_3 y_4 y_5 y_6 x_{C^-}$$

Proof Sketch

Let P_i be a poset for γ_i and $\mathbf{g}_i := \mathbf{g}_{\gamma_i}$. We can express $x_1 x_2$ as

$$\mathbf{x}^{\mathbf{g}_1 + \mathbf{g}_2} \sum_{(I_1, I_2) \in J(P_1) \times J(P_2)} wt(I_1) wt(I_2)$$

where $wt(I)$ is a monomial determined by the content of I. Our primary focus is on finding a partition $J(P_1) \times J(P_2) = A \cup B$ and establishing bijections between A and $J(P_{\gamma_3}) \times J(P_{\gamma_4})$ and between B and $J(P_5) \times J(P_6)$. The next step of our proof will be to show that, $\mathbf{g}_1 + \mathbf{g}_2 = \mathbf{g}_3 + \mathbf{g}_4$, and $\mathbf{g}_1 + \mathbf{g}_2 + \deg_{\mathbf{x}}(Z) = \mathbf{g}_5 + \mathbf{g}_6$ where Z is determined by R and Sw.

Implications

- We immediately can show bracelets and bangles, as in [4], form a spanning set of the cluster algebra from a punctured surface.
- We recover a key statement concerning the linear independence of bracelets and bangles.
- Musiker, Schiffler, and Williams give multiple definitions of a cluster algebra element associated to a notched arc with self-intersection. We show that two of these definitions agree.

Future Directions

- Adapt the poset construction to generalized cluster algebras from orbifolds and study skein relations there.
- Use our skein relations to study extensions in skew-gentle algebras.

References

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