The doubly asymmetric simple exclusion process

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ASEP

The **asymmetric simple exclusion process** (ASEP) is a Markov chain for particles hopping on a one-dimensional lattice such that each site contains at most one particle. The ASEP was introduced independently in biology by [Macdonald-Gibbs-Pipkin], and in mathematics by [Spitzer]. There are many versions of the ASEP: the lattice is not necessarily finite. It can have open boundaries, or be a ring [Liggett]. Particles can have different species/types/colors, and this variation is called the multispecies ASEP (mASEP). We study yet another variation of the mASEP in which q particles with species $1, \dots, p$ hop along a circular lattice with n sites, but also the particles are allowed to spontaneously change from one species to another. This is a discrete analogue of evaporation and deposition.

DASEP

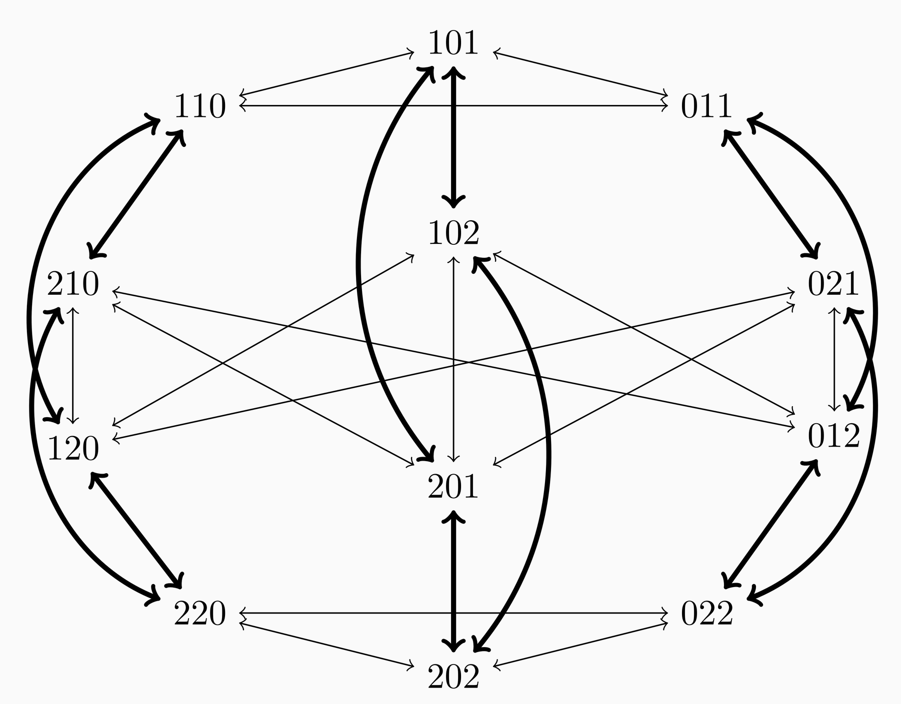

Fig. 1: The state diagram of DASEP(3, 2, 2): 3 sites, 2 species, 2 particles

Let n, p, q be positive integers with $n > q$, and let $u, t \in [0, 1)$ be constants. Introduced by [Ash], the **doubly asymmetric simple exclusion process** DASEP(n, p, q) is a Markov chain on the set of words (or weak compositions) of length n in $0, \dots, p$ with $n - q$ zeros: The transition probability $P(\mu, \nu)$ on two states μ and ν is as follows:

- If $\mu = AijB$ and $\nu = AjiB$ (where A and B are words in $0, \dots, p$) with $i \neq j$, then $P(\mu, \nu) = \frac{t}{3n}$ if $i > j$ and $P(\mu, \nu) = \frac{1}{3n}$ if $j > i$.
- If $\mu = iAj$ and $\nu = jAi$ with $i \neq j$, then $P(\mu, \nu) = \frac{t}{3n}$ if $j > i$ and $P(\mu, \nu) = \frac{1}{3n}$ if $i > j$.
- If $\mu = AiB$ and $\nu = A(i+1)B$ with $i \leq p - 1$, then $P(\mu, \nu) = \frac{u}{3n}$.
- If $\mu = A(i+1)B$ and $\nu = AiB$ with $i \geq 1$, then $P(\mu, \nu) = \frac{1}{3n}$.
- Otherwise $P(\mu, \nu) = 0$ for $\mu \neq \nu$ and $P(\mu, \mu) = 1 - \sum_{\nu \neq \mu} P(\mu, \nu)$.

This Markov chain is irreducible and aperiodic, so it has a unique stationary distribution π given by rational functions in u, t, which satisfies the global balance equations $\pi(\mu) \sum_{\nu \neq \mu} P(\mu, \nu) = \sum_{\nu \neq \mu} \pi(\nu) P(\nu, \mu)$ for any state μ. For convenience, we clear the denominators and obtain the "unnormalized steady state probabilities" π_{DASEP} which are proportional to the stationary distribution by a factor of the *partition function* $Z_n^{p,q} = \sum_{\mu \in \Gamma_n^{p,q}} \pi_{\mathrm{DASEP}}(\mu)$. We require the unnormalized steady state probabilities to be coprime so they are uniquely defined.

Theorem

Consider DASEP(n, p, q) for any positive integers n, p, q with $n > q$.

(1) For any two binary words $w, w' \in \binom{[n]}{q}$, we have $\pi_{\mathrm{DASEP}}(w) = \pi_{\mathrm{DASEP}}(w')$.

(2) For any binary word $w \in \binom{[n]}{q}$ and partition $\lambda = 1^{m_1} 2^{m_2} \cdots p^{m_p}$ with $m_1 + \cdots + m_p = q$, we have

$$\sum_{\mu \in S_n^w(\lambda)} \pi_{\mathrm{DASEP}}(\mu) = u^{|\lambda| - q} \binom{q}{m_1, m_2, \dots, m_p} \pi_{\mathrm{DASEP}}(w).$$

Corollary

Let $t = 1$, then our model is symmetric, dubbed the "doubly symmetric simple exclusion process (DSSEP)". It is a generalization of the model considered by [Salez], which is an exclusion process on a graph (a circle in our case) with a reservoir of particles at each vertex. Recall that $[p+1]_u = 1 + u + \cdots + u^p$ denotes the u analog of the integer $p + 1$. The partition function of DSSEP(n, p, q) is

$$\binom{n}{q} (1 + u + \cdots + u^p)^q = \binom{n}{q} ([p+1]_u)^q.$$

Proof via lumping

The projection map on the state space of the DASEP by forgetting the order of the particles but preserving the position of empty sites is a lumping of the DASEP onto what we call the **colored Boolean process**. Its state space consists of pairs of binary words with q ones and partitions of length q whose largest part is not greater than p:

$$\Omega_n^{p,q} = \{(w, \lambda) | w \in \binom{[n]}{q}, \lambda_1 \leq p, \ell(\lambda) = q\}.$$

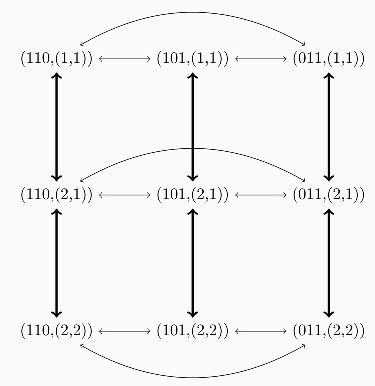

Fig. 2: The state diagram of DASEP(3, 2, 2) after projection

Theorem

Consider the colored Boolean process on n, p, q.

(1) The steady state probabilities of all binary words with the trivial partition are equal, i.e.,

$$\pi_{\mathrm{CBP}}(w, 0^{n-q} 1^q) = \pi_{\mathrm{CBP}}(w', 0^{n-q} 1^q), \quad \text{for all } w, w' \in \binom{[n]}{q}.$$

(2) The steady state probability of an arbitrary state (w, λ) can be expressed in terms of the steady state probability of the corresponding state $(w, 0^{n-q} 1^q)$ with the trivial partition $0^{n-q} 1^q$ as follows:

$$\pi_{\mathrm{CBP}}(w, \lambda) = u^{|\lambda| - q} \binom{q}{m_1, \dots, m_p} \pi_{\mathrm{CBP}}(w, 0^{n-q} 1^q).$$

Theorem

Let $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq -1}$ be polynomial sequences in u, t satisfying the recurrence relation

$$a_k = (u + 2t + 3) a_{k-1} - (t+1)^2 a_{k-2}$$
$$b_k = (u + 2t + 3) b_{k-1} - (t+1)^2 b_{k-2}.$$

with initial conditions $b_{-1} = 0, a_0 = b_0 = 1, a_1 = u + 3t + 4$.

Consider matchings M in the cycle C_{2k+1} or the path L_{2k+1} with $(2k+1)$ vertices. Assign each matching M a weight of $(t+1)^{|M|}(u+1)^{k-|M|}$. Then the stationary distributions of DASEP$(2k+1, 2, 2)$ and DASEP$(2k+2, 2, 2)$ are given by the tables below in which a_k is the generating function of the matchings in C_{2k+1}, and b_k is the generating function of the matchings in L_{2k+1}, i.e.,

$$a_k = \sum_{M : C_{2k+1}} (t+1)^{|M|} (u+1)^{k-|M|}$$
$$b_k = \sum_{M : L_{2k+1}} (t+1)^{|M|} (u+1)^{k-|M|}.$$

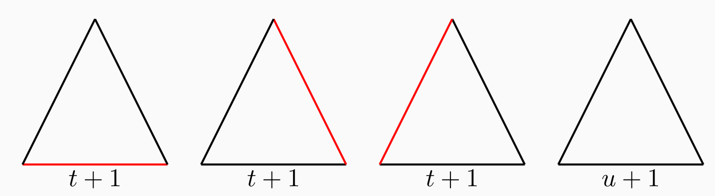

Fig. 3: $a_1 = u + 3t + 4$

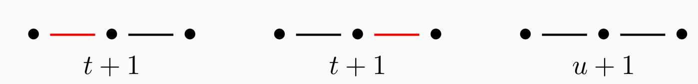

Fig. 4: $b_1 = u + 2t + 3$

μ	$\pi_{\mathrm{DASEP}(2k+1,2,2)}(\mu)$
$S_n((1,1,0,\dots,0))$	a_k
$0\dots010^m20\dots0$	$ua_k + u(t-1)(t+1)^m a_{k-m-1}, (0 \leq m < k)$
$0\dots020^m10\dots0$	$ua_k - u(t-1)(t+1)^m a_{k-m-1}, (0 \leq m < k)$
$S_n((2,2,0,\dots,0))$	$u^2 a_k$

μ	$\pi_{\mathrm{DASEP}(2k+2)}(\mu)$
$S_n((1,1,0,\dots,0))$	b_k
$0\dots010^m20\dots0$	$ub_k + u(t-1)(t+1)^m b_{k-m-1}, (0 \leq m \leq k)$
$0\dots020^m10\dots0$	$ub_k - u(t-1)(t+1)^m b_{k-m-1}, (0 \leq m \leq k)$
$S_n((2,2,0,\dots,0))$	$u^2 b_k$

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