Inhomogeneous particle process defined by canonical Grothendieck polynomials

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1. Introduction

We construct a time, particle, and position inhomogeneous discrete time particle process on the nonnegative integers that generalizes one of those studied in a Dieker and Warren [DW08]. The particles move according to an inhomogeneous geometric distribution and stay in (weakly) decreasing order, where smaller particles block larger particles. We show that the transition probabilities for our particle process is given by a (refined) canonical Grothendieck function up to a simple overall factor.

2. Grothendieck polynomial

Let $\mathcal{P}$ denote the set of all **partitions** $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, drawn in English convention, with $\ell(\lambda) = \max_\ell(\lambda_\ell > 0) < \infty$ being the **length** of λ. A **hook** is a partition $a1^m$ with **arm** $a-1$ and **leg** m. $\mathbf{x}_n := (x_1, \dots, x_n, 0, 0, \dots)$ indeterminates. We take parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots)$.

Definition 1 *A* **hook-valued tableau** *of shape λ is a filling of the Young diagram by hook shaped tableau, fillings of a hook shape with entries weakly (resp. strictly) increasing along the arm (resp. leg), satisfying the local conditions*

$$\begin{array}{|c|c|}\hline \mathsf{a} & \mathsf{b} \\ \hline \mathsf{c} & \\ \cline{1-1} \end{array} \qquad \begin{array}{c} \max(\mathsf{a}) \leq \min(\mathsf{b}) \\ \wedge \\ \min(\mathsf{c}) \end{array}$$

Definition 2 *The (refined)* **canonical Grothendieck function** *is the generating function*

$$G_\lambda(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_T \prod_{\mathsf{b} \in T} (-\alpha_i)^{a(\mathsf{b})} (-\beta_j)^{b(\mathsf{b})} x_1^{\#1(\mathsf{b})} \cdots x_n^{\#n(\mathsf{b})},$$

where we sum over all hook-valued tableaux T of shape λ, product over all entries b in T with $a(\mathsf{b})$ (resp. $b(\mathsf{b})$) the arm (resp. leg) of the shape of b and i (resp. j) the row (resp. column) of b.

The set $\{G_\lambda(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta})\}_{\lambda \in \mathcal{P}}$ is a basis for symmetric functions (see, *e.g.*, [HJKSS24]). We can define the skew canonical Grothendiecks $G_{\lambda /\!\!/ \mu}$ by $G_{\lambda /\!\!/ \emptyset} = G_\lambda$ and the branching rule

$$G_{\lambda /\!\!/ \mu}(\mathbf{x}_n, \mathbf{y}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\mu \subseteq \nu \subseteq \lambda} G_{\lambda /\!\!/ \nu}(\mathbf{y}_m; \boldsymbol{\alpha}, \boldsymbol{\beta}) G_{\nu /\!\!/ \mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}). \quad (1)$$

Remark 3 *This is not the natural skew shape definition $G_{\lambda/\mu}$. (See [IMS24, Sec. 4.1]).*

3. TASEP

We consider additional parameters $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$. Let $G(j, i)$ be the position of the j-th particle at time i given by

$$G(j, i) = \min\bigl(G(j, i-1) + w_{ji}, G(j-1, i-1)\bigr), \quad (2)$$

($G(0, i-1) := \infty$), where the random variable w_{ij} is determined by the **inhomogeneous geometric distribution** (which depends on $G(j, i-1)$) defined as

$$\mathsf{P}_{\mathcal{G}}(w_{ji} = m' \mid G(j, i-1) = m) := \frac{1 - \pi_j x_i}{1 + \alpha_{m+m'} x_i} \prod_{k=m}^{m+m'-1} \frac{(\alpha_k + \pi_j) x_i}{1 + \alpha_k x_i}. \quad (3)$$

In other words, the j-th particle at time i attempts to jump w_{ji} steps, but can be blocked by the $(j-1)$-th particle, which updates its position after the j-th particle moves.

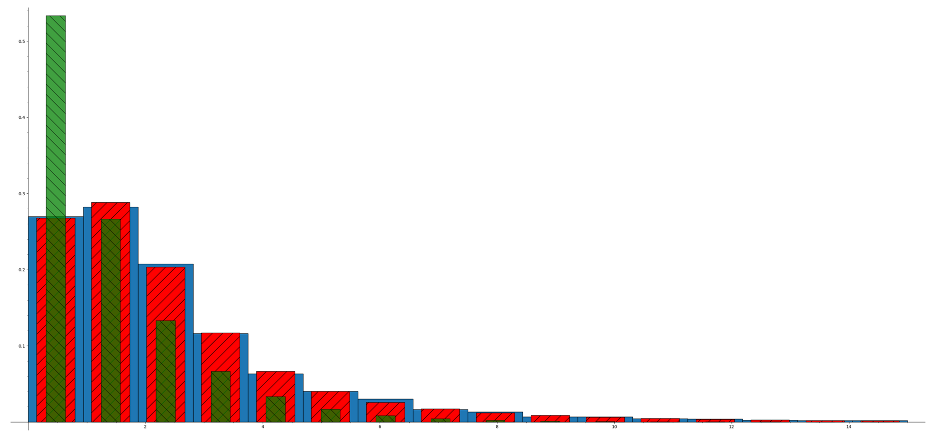

Figure 1: *A sampling of 10000 samples of the inhomogeneous geometric distribution $\mathsf{P}_{\mathcal{G}}$ for $x_i = 1$, $\pi_j = .5$, and $\alpha_k = 1 - ke^{-k/2}$ (blue), compared with the exact distribution (red) and the geometric distribution with parameter $\pi_j x_i$ (green).*

The $\boldsymbol{\alpha}$ parameters act as a current being applied to the system, where the strength (and direction) can vary at each position. When $\boldsymbol{\alpha} < 0$, the $\boldsymbol{\alpha}$ acts as (position-based) viscosity. Locations where certain particles must stop can be given by $-\alpha_k = \pi_j$.

Theorem 4 ([IMS23]) *Suppose $\ell(\lambda) \leq \ell$, $\pi_j x_i \in (0,1)$, $\alpha_k x_i > -1$, and $\alpha_k + \pi_j \geq 0$ for all i, j, k. Set $\beta_j = \pi_{j+1}$. Let $\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)$ denote the n-step transition probability for the particle system using the distribution* (3) *for the jump probability of the particles with interactions given by* (2)*. Then,*

$$\mathsf{P}_{\mathcal{C},n}(\lambda|\mu) = \prod_{i=1}^n (1 - \pi_1 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{\lambda/\mu} G_{\lambda /\!\!/ \mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}),$$

where $(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{\lambda/\mu} := \prod_{(i,j) \in \lambda/\mu} (\alpha_{i-1} + \pi_j)$. (The α_0 may not be 0.)

Remark 5 *Our TASEP at $\boldsymbol{\beta} = 0$ is the same model as a special case of [KPS19] (with a shift to fermionic indexing).*

We can similarly define a Bernoulli process with the position-dependent probability with rates $(\rho_1, \rho_2, \dots)$ by

$$\mathsf{P}_{\mathcal{B}}(w_{ji} = 1 \mid G(j, i-1) = m) := \frac{(\rho_j + \beta_m) x_i}{1 + \rho_j x_i}. \quad (4)$$

Theorem 6 ([IMS23]) *Suppose $\lambda_1 \leq \ell$, $\beta_k x_i \in (0,1)$, $\rho_j x_i > -1$, and $\rho_j + \beta_k \geq 0$ for all i, j, k. Set $\alpha_j = \rho_{j+1}$. The n-step transition probability for the particle system using Bernoulli jumps according to the distribution* (4) *is given by*

$$\mathsf{P}_{\mathcal{B},n}(\lambda|\mu) = \frac{(\vec{\boldsymbol{\beta}} + \boldsymbol{\rho})^{\lambda/\mu}}{\prod_{i=1}^n (1 + \rho_1 x_i)} G_{\lambda' /\!\!/ \mu'}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}).$$

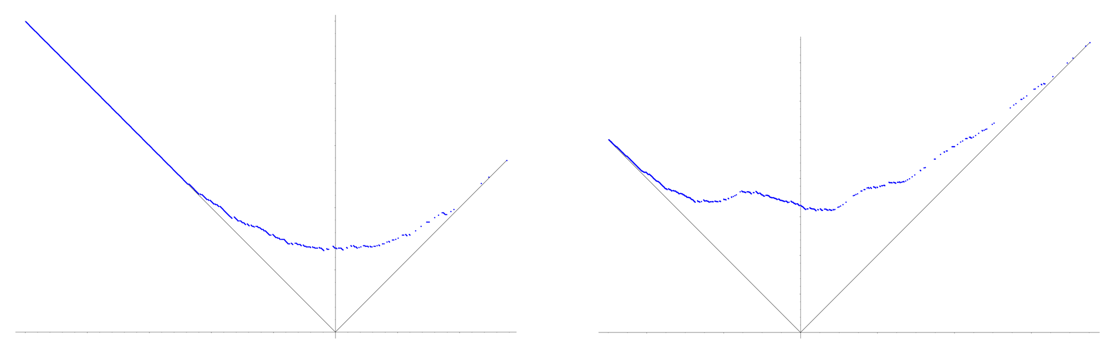

Figure 2: *Samples of our process with $\ell = 500$ particles after $n = 50000$ time steps with (left) $\boldsymbol{\pi} = 1$, $\mathbf{x} = 0.01$, and $\boldsymbol{\alpha} = -0.5$; (right) $\boldsymbol{\pi} = 0.5$, $\mathbf{x} = .2$, and $\alpha_k = 0.5 \sin(k/50)^6$.*

4. Noncommutative operators

The **Schur operator** $\kappa_i \colon \mathbf{k}[\mathcal{P}] \to \mathbf{k}[\mathcal{P}]$ adds a box to the i-th row of λ if possible and is 0 otherwise. Define the linear operator

$$U_i := \kappa_i + \Theta_i, \quad \text{where} \quad \Theta_i \cdot \lambda := \begin{cases} -\alpha_{\lambda_i} \lambda & \text{if } \lambda_i < \lambda_{i-1}, \\ \beta_{i-1} \lambda & \text{if } \lambda_i = \lambda_{i-1}, \end{cases} \quad (5)$$

Lemma 7 ([IMS24]) $\mathbf{U} = \{U_i\}_{i=1}^\infty$ *satisfy the weak Knuth relations*

To relate our TASEP with $\mathbf{U}$, we use the basis $\{{}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|\}_{\lambda \in \mathcal{P}}$ of the (dual) fermion Fock space $\mathcal{F}^\dagger$ such that ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu| e^{H(\mathbf{x}_n)} = \sum_\lambda G_{\lambda /\!\!/ \mu}(\mathbf{x}_n; \boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|$, where $e^{H(\mathbf{x}_n)}$ is a vertex operator acting on $\mathcal{F}^\dagger$. Assume $\alpha_0 = 0$ for simplicity. Via the skew Cauchy formula [IMS24, HJKSS24] with particular specializations and combinatorial description [HJKSS24], we obtain

$${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu| e^{H(\mathbf{x}_n)} = \prod_{i=1}^n (1 - \pi_1 x_i)^{-1} \sum_{\lambda \supseteq \mu} \frac{\mathsf{P}_{\mathcal{C},n}(\lambda|\mu)}{(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{\lambda/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|. \quad (6)$$

($\beta_j = \pi_{j+1}$). We restrict to a single timestep at time i to encode the growth process by $\mathbf{U}$. Since we have a Markov process, $\mathsf{P}_{\mathcal{C},n+n'}(\lambda|\mu) = \sum_\nu \mathsf{P}_{\mathcal{C},n}(\lambda|\nu) \mathsf{P}_{\mathcal{C},n'}(\nu|\mu)$, which agrees with the branching rules (1). Define the time evolution operator

$$\mathcal{T}_{\mathcal{C}} := \sum_{k=0}^\infty h_k(x_i \mathbf{U}) = \sum_{k=0}^\infty x_i^k h_k(\mathbf{U}), \quad (7)$$

where $h_k(\mathbf{U})$ is the noncommutative complete symmetric function. By some plethystic manipulations as in [IMS23, Sec. 4.2],

$${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\mu| e^{H(x_i)} = \prod_{j=2}^\infty (1 - \pi_j x_i)^{-1} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu|.$$

Thus, if we consider the expansion ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu| = \sum_\lambda B_{\lambda\mu} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle\lambda|$, and matching coefficients in (6), we obtain the one-step transition probability at time i:

$$\mathsf{P}_{\mathcal{C}}(\lambda|\mu) = \frac{B_{\lambda\mu}}{(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{\lambda/\mu}} \prod_{j=1}^\infty (1 - \pi_j x_i)^{-1}.$$

We could also prove Theorem 4 by using the combinatorics of hook-valued tableaux as in [IMS23, Sec. 5.3], where the positions of the particles is dictated by the smallest value in each entry of the hook-valued tableaux. The key observation is that we have a factor $x_i(1 - \alpha_k x_i)^{-1}$ for every box in the k-th column that would normally contain an i in the set-valued tableaux (over all k), or where there is no arm. The leg (the column part except for the corner) corresponds to the choice between 1 and $-\pi_i x_j$ in the numerator of the normalization constant as in [IMS23, Sec. 5.3].

5. One time step example

Example 8 *Let $\mu = (1,1)$, $\alpha_0 = 0$, and $\pi_j = 0$ for all $j > 3$. As*

$$\begin{aligned} h_1(\mathbf{u}_3) &= u_1 + u_2 + u_3, \\ h_2(\mathbf{u}_3) &= u_1^2 + u_1 u_2 + u_1 u_3 + u_2^2 + u_2 u_3 + u_3^2, \\ h_3(\mathbf{u}_3) &= u_1^3 + u_1^2 u_2 + u_1^2 u_3 + u_1 u_2^2 + u_1 u_2 u_3 \\ &\quad + u_1 u_3^2 + u_2^3 + u_2^2 u_3 + u_2 u_3^2 + u_3^3, \end{aligned}$$

we compute

$$\begin{aligned} h_1(\mathbf{U}_3) \cdot \mu &= \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \beta_1 \ydiagram{1,1} + \ydiagram{1,1,1}, \\ h_2(\mathbf{U}_3) \cdot \mu &= \left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1 + \alpha_2) \ydiagram{2,1} + \ydiagram{3,1}\right) + \beta_1 \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) \\ &\quad + \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) + \beta_1^2 \ydiagram{1,1} + \beta_1 \ydiagram{1,1,1} + \beta_2 \ydiagram{1,1,1}, \\ h_3(\mathbf{U}_3) \cdot \mu &= \left(-\alpha_1^3 \ydiagram{1,1} + h_2(\alpha_1, \alpha_2) \ydiagram{2,1} - h_1(\alpha_1, \alpha_2, \alpha_3) \ydiagram{3,1} + \ydiagram{4,1}\right) \\ &\quad + \beta_1 \left(\alpha_1^2 \ydiagram{1,1} - (\alpha_1 + \alpha_2) \ydiagram{2,1} + \ydiagram{3,1}\right) + \left(\alpha_1^2 \ydiagram{1,1,1} - (\alpha_1 + \alpha_2) \ydiagram{2,1,1} + \ydiagram{3,1,1}\right) \\ &\quad + \beta_1^2 \left(-\alpha_1 \ydiagram{1,1} + \ydiagram{2,1}\right) + \beta_1 \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) + \beta_2 \left(-\alpha_1 \ydiagram{1,1,1} + \ydiagram{2,1,1}\right) \\ &\quad + \beta_1^3 \ydiagram{1,1} + \beta_1^2 \ydiagram{1,1,1} + \beta_1 \beta_2 \ydiagram{1,1,1} + \beta_2^2 \ydiagram{1,1,1}. \end{aligned}$$

Let $A_k = -\boldsymbol{\alpha}_k = \{-\alpha_1, \dots, -\alpha_k\}$. Therefore, we have

$$\begin{aligned} {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu| &= (1 + h_1(\boldsymbol{\beta}_1 \sqcup A_1) x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_1) x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1, 1| \\ &\quad + x_i(1 + h_1(\boldsymbol{\beta}_1 \sqcup A_2) x_i + h_2(\boldsymbol{\beta}_1 \sqcup A_2) x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2, 1| \\ &\quad + x_i(1 + h_1(\boldsymbol{\beta}_2 \sqcup A_1) x_i + h_2(\boldsymbol{\beta}_2 \sqcup A_1) x_i^2 + \cdots) \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1, 1, 1| + \cdots \end{aligned}$$

Using $\beta_j = \pi_{j+1}$, ${}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle \mathcal{T}_{\mathcal{C}} \cdot \mu|$ is simplified as

$$\begin{aligned} &\frac{(1 + \alpha_1 x_i)^{-1}}{1 - \pi_2 x_i} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1, 1| + \frac{(\alpha_1 x_i + \pi_1 x_i)(1 + \alpha_1 x_i)^{-1}(1 + \alpha_2 x_i)^{-1}}{(1 - \pi_2 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(2,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 2, 1| \\ &\quad + \frac{\pi_3 x_i (1 + \alpha_1 x_i)^{-1}}{(1 - \pi_2 x_i)(1 - \pi_3 x_i)(\vec{\boldsymbol{\alpha}} + \boldsymbol{\pi})^{(1,1,1)/\mu}} \cdot {}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]}\langle 1, 1, 1| + \cdots. \end{aligned}$$

Seeing coefficients, we obtain the one-step transition probabilities at time i:

$$\begin{aligned} \mathsf{P}_{\mathcal{C}}(1, 1|\mu) &= \frac{(1 - \pi_1 x_i)(1 - \pi_3 x_i)}{(1 + \alpha_1 x_i)}, \\ \mathsf{P}_{\mathcal{C}}(2, 1|\mu) &= \frac{(\alpha_1 x_i + \pi_1 x_i)(1 - \pi_1 x_i)(1 - \pi_3 x_i)}{(1 + \alpha_1 x_i)(1 + \alpha_2 x_i)}, \\ \mathsf{P}_{\mathcal{C}}(1, 1, 1|\mu) &= \frac{\pi_3 x_i (1 - \pi_1 x_i)}{(1 + \alpha_1 x_i)}, \quad \text{etc.} \end{aligned}$$

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