Levi-spherical varieties and Demazure characters

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Abstract

We prove a short, root-system uniform, combinatorial classification of Levi-spherical Schubert varieties for any generalized flag variety G/B of finite Lie type. We apply this to the study of multiplicity-free decompositions of Demazure modules and their characters.

Schubert varieties

Let G be a complex, connected, reductive algebraic group of rank r. Fix a maximal torus T in G, and a Borel subgroup B of G containing T. The homogeneous space G/B is a smooth projective variety known as the *full flag variety*. The study of the flag variety first arose out of the need to formalize and justify the enumerative geometry of H. Schubert as laid out in Hilbert's 15th problem.

The B-orbits for the action of B by left translation yield a cellular filtration of G/B; these orbits, denoted X_w°, are referred to as *Schubert cells* and are indexed by elements w of the Weyl group W of G. C. Chevalley introduced the now ubiquitous Bruhat order to describe the inclusion order of B-orbit closures in G/B. These orbit closures, the *Schubert varieties* X_w for $w \in W$, are well-studied varieties that boast a rich combinatorial structure that encodes many facets of their geometry.

Spherical varieties

If an algebraic group H acts on a variety X by a morphism of algebraic varieties, we say that X is an H-variety. Let H be a reductive algebraic group and B_H a Borel subgroup of H. The *H-complexity* of an H-variety X, denoted $c_H(X)$, is the minimum codimension of a B_H-orbit in X.

The normal H-varieties with H-complexity equal to 0 are the *H-spherical varieties*. Spherical varieties generalize several important classes of algebraic varieties including toric varieties, projective rational homogeneous spaces and symmetric varieties.

Levi subgroup actions

Our choice of T and B determine the *root system* Φ and *simple roots* $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, respectively. The Weyl group W of G, is generated by the set of simple reflections $\{s_i := s_{\alpha_i} | \alpha_i \in \Delta\}$. For $I \subseteq \Delta$, W_I is the subgroup of W generated by $\{s_{\alpha_i} | \alpha_i \in I\}$. The *standard parabolic subgroups* of G containing B are indexed by subsets of Δ. The standard parabolic subgroup associated to I is $P_I := BW_IB$ with Levi decomposition

$$P_I = L_I \ltimes U_I,$$

where L_I is a reductive subgroup called a *Levi subgroup*, and U_I is the unipotent radical of P_I. Define $B_{L_I} := L_I \cap B$. Then B_{L_I} is a Borel subgroup of L_I, and we shall refer to such subgroups as *Levi-Borel subgroups*.

While B_{L_I} acts on any X_w, since $B_{L_I} \subseteq B$, the same is not true for L_I. The set of *left descents* of w is

$$\mathcal{D}_L(w) = \{\beta \in \Delta : \ell(s_\beta w) < \ell(w)\},$$

where $\ell(w) = \dim X_w$ is the *Coxeter length* of w. The stabilizer of X_w in G for the action by left translation is the standard parabolic subgroup $P_{\mathcal{D}_L(w)}$. The Levi subgroups $L_I \leq P_I \leq P_{\mathcal{D}_L(w)}$ for $I \subseteq \mathcal{D}_L(w)$ are a family of reductive algebraic groups acting on X_w.

Spherical elements

A *standard Coxeter element* $c \in W_I$ is any product of all of the generators of W_I in some order. Let $w_0(I)$ be the longest element of W_I. The following definition was given in [1, Definition 1.1].

Definition

Let $w \in W$ and $I \subseteq \mathcal{D}_L(w)$. Then w is *I-spherical* if $w_0(I)w$ is a standard Coxeter element for W_J where $J \subseteq \Delta$.

Classification

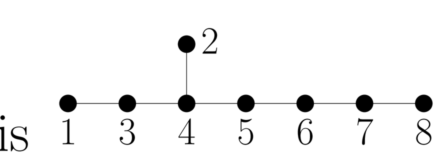
Theorem 1 ([2])

Fix $w \in W$ and $I \subseteq \mathcal{D}_L(w)$. X_w is L_I-spherical if and only if w is I-spherical.

This theorem resolves the main conjecture of [3]. In [1], this theorem was established for $G = GL_n$.

Examples

The E_8 Dynkin diagram is

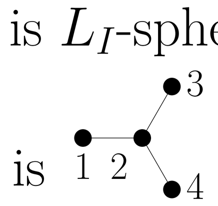


. One associates the simple roots α_i $(1 \leq i \leq 8)$ with this labeling and writes $s_i := s_{\alpha_i}$. Suppose

$$w = s_2s_3s_4s_2s_3s_4s_5s_4s_2s_3s_1s_4s_5s_6s_7s_6s_8s_7s_6 \in W.$$

Then $\mathcal{D}_L(w) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8\}$. Let $I = \mathcal{D}_L(w)$. Since $w_0(I)w = s_1s_6s_7s_8$ is a standard Coxeter element X_w is L_I-spherical.

The D_4 diagram is



. Let

$$w = s_3s_2s_3s_4s_2s_1s_2 \in W.$$

Set $I = \mathcal{D}_L(w) = \{\alpha_2, \alpha_3\}$. Thus $w_0(I) = s_2s_3s_2$ and $w_0(I)w = s_4s_2s_1s_2$ is not standard Coxeter. Hence X_w is not L_I-spherical.

Demazure modules

Let $\mathfrak{X}(T)$ denote the lattice of weights of T; our fixed Borel subgroup B determines a subset of dominant integral weights $\mathfrak{X}(T)^+ \subset \mathfrak{X}(T)$. For $\lambda \in \mathfrak{X}(T)^+$, let $\mathfrak{L}_\lambda$ be the associated line bundle on G/B. For $w \in W$, we write $\mathfrak{L}_\lambda|_{X_w}$ for the restriction of $\mathfrak{L}_\lambda$ to the Schubert subvariety $X_w \subseteq G/B$. Then the *Demazure module* V_λ^w is isomorphic to the dual of the space of global sections of $\mathfrak{L}_\lambda|_{X_w}$, that is

$$V_\lambda^w \cong H^0(X_w, \mathfrak{L}_\lambda|_{X_w})^*.$$

This geometric perspective highlights the fact that V_λ^w is not just a B-module, but is also a L_I-module.

Demazure characters

Let $\mathfrak{X}_{L_I}(T)^+$ be the set of dominant integral weights with respect to the choice of maximal torus and Borel subgroup $T \subseteq B_I \subseteq L_I$. For $\mu \in \mathfrak{X}_{L_I}(T)^+$, let $V_{L_I,\mu}$ be the associated irreducible L_I-module. If M is a L_I-module and

$$M = \bigoplus_{\mu \in \mathfrak{X}_{L_I}(T)^+} V_{L_I,\mu}^{\oplus m_{L_I,\mu}}$$

is the decomposition into irreducible L_I-modules, then M is a *multiplicity-free L_I-module* if $m_{L_I,\mu} \in \{0,1\}$. Similarly, if $\mathrm{char}(M)$ is the formal T-character of M then

$$\mathrm{char}(M) = \sum_{\mu \in \mathfrak{X}_{L_I}(T)^+} m_{L_I,\mu} \mathrm{char}(V_{L_I,\mu}),$$

is *I-multiplicity-free* if $m_{L_I,\mu} \in \{0,1\}$.

Theorem 2 ([2])

Let $w \in W$ be I-spherical for $I \subseteq D_L(w)$. For all $\lambda \in \mathfrak{X}(T)^+$, the Demazure character $\mathrm{char}(V_\lambda^w)$ is I-multiplicity-free.

References

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[2] Yibo Gao, Reuven Hodges, and Alexander Yong. "Levi-Spherical Schubert varieties". *Advances in Mathematics* 439 (2024).

[3] Reuven Hodges and Alexander Yong. "Coxeter combinatorics and Spherical Schubert geometry". *Journal of Lie Theory* 32.2 (2022), pp. 447–474.

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