A Whitney polynomial for hypermaps

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Hypermaps and their Whitney polynomial

A *collection of hypermaps* is a pair of permutations (σ, α) acting on the same finite set of labels. The orbits of this action are the connected components, we denote their number by $\kappa(\sigma,\alpha)$. We call (σ,α) a *hypermap* if $\kappa(\sigma,\alpha)=1$. Fig. 1 represents the hypermap (σ,α) for $\sigma=(1,4)(2,5)(3)$ and $\alpha=(1,2,3)(4,5)$. The cycles of σ are the *vertices*, the cycles of α are the *hyperedges* and the cycles of $\alpha^{-1}\sigma=(1,5)(2,4,3)$ are the *faces*. A hypermap is a *map* if the length of each cycle in α is at most 2.

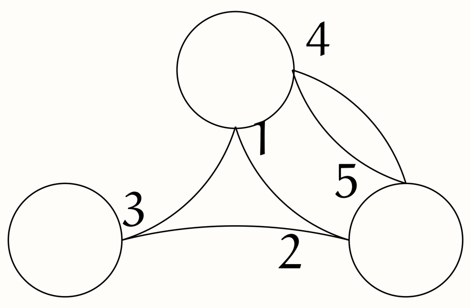


Figure 1: The hypermap (σ, α)

The *Whitney polynomial* $R(\sigma,\alpha;u,v)$ of a collection of hypermaps (σ,α) on a set of n points is defined by the formula

$$R(\sigma,\alpha;u,v)=\sum_{\beta\leq\alpha} u^{\kappa(\sigma,\beta)-\kappa(\sigma,\alpha)}\cdot v^{\kappa(\sigma,\beta)+n-z(\beta)-z(\sigma)}$$

Here the summation is over all permutations β refining α, and $z(\pi)$ denotes the number of cycles of π. For maps we recover the usual definition of the Whitney polynomial of the underlying graph.

General properties of the Whitney polynomial

Assume $H=(\sigma,\alpha)$ is a collection of hypermaps and $(1,2,\ldots,m)$ is a cycle of α for some $m\geq 2$. Then we have the following "deletion-contraction rule":

$$R(H;u,v)=\sum_{k=1}^{m} R(\phi_k(H);u,v)\cdot w_k,$$

Here $\phi_k(H)=(\sigma,(1,k)\alpha(1,k-1))$ and $w_k=u^{\kappa(\phi_k(H))-\kappa(H)}v$ if $k\neq 1$ but 1 and k belong to the same cycle of σ, and $\phi_k(H)=((1,k)\sigma,(1,k)\alpha(1,k-1))$ and $w_k=u^{\kappa(\phi_k(H))-\kappa(H)}$ in all other cases. Note that each w_k is a monomial from the set $\{1,u,v,uv\}$.

Noteworthy specializations of the Whitney polynomial include:

1. $R(\sigma,\alpha;0,0)$ is the number of spanning hyperforests of (σ,α).
2. $R(\sigma,\alpha;0,1)$ is the number of spanning collections of hypermaps of (σ,α).

The Tutte polynomial $T(G;x,y)$ of a graph (map) G is given by $T(G;x,y)=R(G;x-1,y-1)$. Extending this definition to collections of hypermaps the obvious way does not seem to be a good idea:

Example For $\sigma=(1)(2)\cdots(n)$ and $\alpha=(1,2,\ldots,n)$ the polynomial $R(\sigma,\alpha;u,v)$ is a *Narayana polynomial* of u For $n=2$ we have $R(\sigma,\alpha;u,v)=1+u$, but for $n=3$ we get $R(\sigma,\alpha;u,v)=u^2+3u+1$: substituting $u=x-1$ yields x^2+x-1.

We still have the following generalized duality result.

Theorem *A collection of hypermaps (σ,α) of genus zero and its dual collection $(\alpha^{-1}\sigma,\alpha^{-1})$ satisfy* $R(\sigma,\alpha;u,v)=R(\alpha^{-1}\sigma,\alpha^{-1};v,u)$.

Medial maps

Definition Let (σ,α) be a collection of hypermaps on the set of points $\{1,2,\ldots,n\}$. We define its *medial map* $M(\sigma,\alpha)$ as the following map (σ',α') on $\{1^-,1^+,2^-,2^+,\ldots,n^-,n^+\}$:

1. the cycles of σ' are all cycles of the form $(i_1^-,i_1^+,i_2^-,i_2^+,\ldots,i_k^-,i_k^+)$ where $(i_1,i_2,\ldots,i_k)$ is a cycle of α;
2. the cycles of α' are all cycles of the form $(i^+,\sigma(i)^-)$.

We obtain a collection of maps (σ',α') which has an *underlying Eulerian digraph* obtained by directing each edge (i^+,j^-) as $i^+\to j^-$.

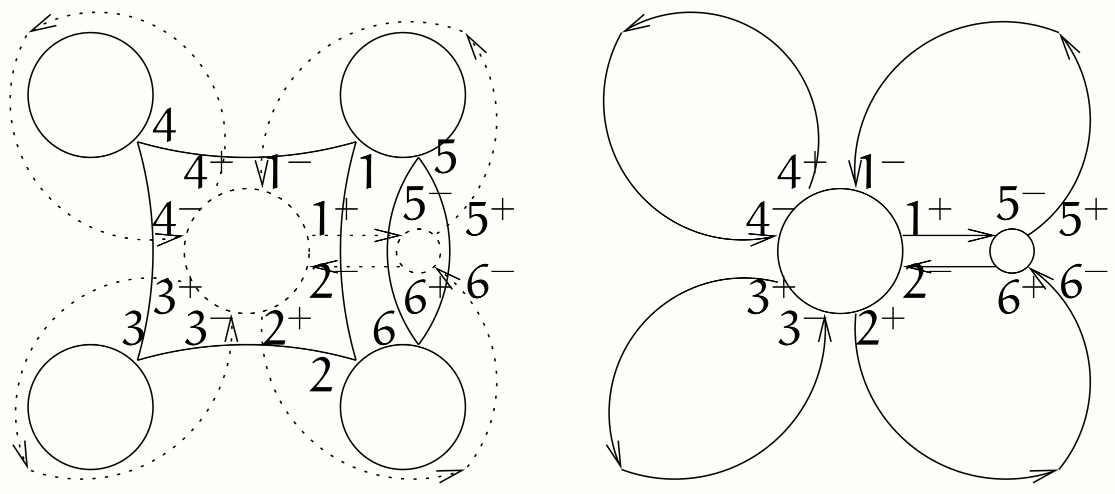


Figure 2: A planar hypermap and its medial map

Proposition *Every directed Eulerian graph arises as the directed medial graph of a collection of hypermaps.*

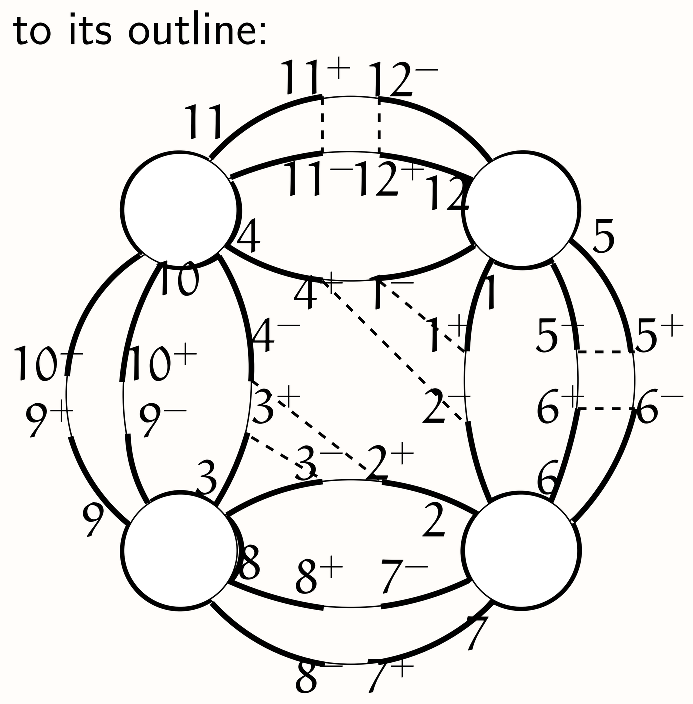
A *noncrossing Eulerian state* is a partitioning of the edges of the underlying directed medial graph into closed paths in such a way that these paths do not cross at any of the vertices. The *noncrossing circuit partition polynomial* of an Eulerian map (σ,α) as $j((\sigma,\alpha);x)=\sum_{k\geq 0} f_k(\sigma,\alpha)x^k$. Here $f_k(\sigma,\alpha)$ is the number of noncrossing Eulerian states with k cycles.

The following result generalizes Ellis-Monaghan's generalization of Martin's formula [1, Eq. (15)] from planar maps to hypermaps.

Theorem *Let (σ,α) be a genus zero collection of hypermaps and $M(\sigma,\alpha)$ the collection of its medial maps. Then $j(M(\sigma,\alpha);x)=x^{\kappa(\sigma,\alpha)}R(\sigma,\alpha;x,x)$ holds.*

A visual computation of $R(\sigma,\alpha;u,v)$ in the planar case

Given a planar hypermap, we may shrink the edges of its medial map to its outline:



We cut the hyperedges connecting positive points of the medial map to negative points. We assume that the infinite outer face is a sea, and its water flows in along the cuts:

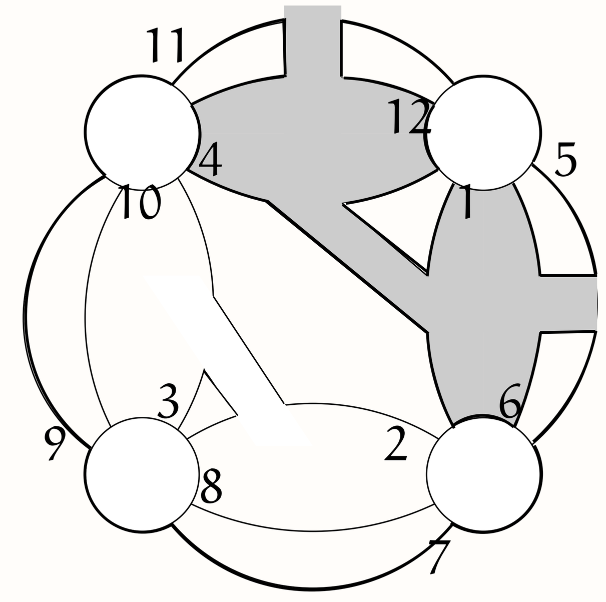


Figure 3: Wet coastlines after a few cuts

Theorem *Given a planar hypermap (σ,α), we may visually compute its Whitney polynomial by making its model in paper, and performing the above cutting procedure in all possible ways and associating to each outcome u raised to the power of the wet coastlines and v raised to the power of the dry faces. The sum of all weights is $u\cdot R(\sigma,\alpha;u,v)$.*

The characteristic polynomial and the flow polynomial

Proposition *Let α be a permutation of $\{1,2,\ldots,n\}$ with k cycles of lengths $c_1,\ldots,c_k$. Then the partially ordered set of all refinements of α is the direct product $[\mathrm{id},\alpha]=\prod_{i=1}^{k}\mathrm{NC}(c_i)$. Here id is the identity permutation and $\mathrm{NC}(c_i)$ is the lattice of noncrossing partitions on c_i elements.*

We define the *characteristic polynomial* $\chi(\sigma,\alpha;t)$ by

$$\chi(\sigma,\alpha;t)=\sum_{\beta\leq\alpha}\mu(\mathrm{id},\beta)\cdot t^{\kappa(\sigma,\beta)-\kappa(\sigma,\alpha)}.$$

Theorem *Assume that each cycle of α has length at most 3. Then for any positive integer t, the number $t^{\kappa(\sigma,\alpha)}\cdot\chi(\sigma,\alpha,t)$ is the number of ways to t-color the vertices of (σ,α) in such a way that no two vertices of the same color are incident to the same cycle of α.*

We define the *flow polynomial* $C(\sigma,\alpha;t)$ of (σ,α) by

$$C(\sigma,\alpha;t)=\sum_{\beta\leq\alpha}\mu(\beta,\alpha)t^{n+\kappa(\sigma,\beta)-z(\beta)-z(\sigma)}.$$

Following Cori and Machì [2] we define a *flow* on a (σ,α) as a function $f:\{1,2,\ldots,n\}\to K$ for some field K such that the set C of points of any cycle of σ or α satisfies $\sum_{i\in C}f(i)=0$.

Theorem *Let K be a finite field with q elements and assume that each cycle of α has at most three elements. Then $C(\sigma,\alpha;q)$ is the number of nowhere zero flows $f:\{1,2,\ldots,n\}\to K$ on (σ,α).*

References

[1] J. A. Ellis-Monaghan, Identities for circuit partition polynomials, with applications to the Tutte polynomial, Special issue on the Tutte polynomial, *Adv. in Appl. Math.* **32** (2004), 188–197.

[2] R. Cori and A. Machì, Flows on hypermaps, *Glasgow Math. J.* **30** (1988), 17–29.