Extremal weight crystals over affine Lie algebras of infinite rank

Taehyeok Heo† Seoul National University
gjxogur123@snu.ac.kr

1. Introduction

Extremal weight crystal

· $V(\lambda)$ $(\lambda \in P)$: extremal weight module
$= U_q(\mathfrak{g})$-module generated by an extremal weight vector v_λ
Remark When $\lambda \in P^+$, $V(\lambda)$ is a highest weight module.

· $B(\lambda)$: crystal base of $V(\lambda)$ [Kashiwara 94]
= local basis of $V(\lambda)$ at $q = 0$
- It contains combinatorial data of underlying modules.
- There are many well-known combinatorial models to describe $B(\lambda)$

Motivation [Kwon 11, Naito-Sagaki 12] $(\mathfrak{g} = \mathfrak{a}_\infty, \mathfrak{b}_\infty, \mathfrak{c}_\infty, \mathfrak{d}_\infty)$
For a nonnegative level $\lambda \in P$, there exist $\lambda^0 \in E$ and $\lambda^+ \in P^+$ such that

$$B(\lambda) \cong B(\lambda^0) \otimes B(\lambda^+)$$

Goal
1. Define an extremal weight crystal structure on combinatorial models;
– spinor model – $\mathfrak{g}_\infty$-type Kashiwara-Nakashima tableaux
2. Describe the algebraic structure of the Grothendieck ring related to extremal weight crystals (using tensor product decompositions)

※ *To illustrate explicitly, we describe only the case for type C from now on.*

2. Notations

· $\mathfrak{g} = \mathfrak{g}_\infty$: affine Lie algebras of infinite rank

$\mathfrak{c}_\infty$:

- $I = \mathbb{Z}_+$: index set (nonnegative integers)
- $\{\alpha_i \mid i \in I\}$: simple roots
  ex) $\alpha_0 = -2\epsilon_1$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(i \geqslant 1)$
- $\{\Lambda_i^{\mathfrak{g}} \mid i \in I\}$: fundamental weights
  ex) $\Lambda_i^{\mathfrak{g}} = \Lambda_0^{\mathfrak{g}} + (\epsilon_1 + \cdots + \epsilon_i)$ $(i \geqslant 1)$
- $P = \mathbb{Z}\Lambda_0^{\mathfrak{g}} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i$: weight lattice
- P^+ : dominant weights
- $E = \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i \subseteq P$

· $\mathcal{P}$: the set of partitions
– $\ell(\lambda)$: the length of $\lambda \in \mathcal{P}$
– $\mathscr{P}(G) = \{ (\lambda, \ell) \in \mathcal{P} \times \mathbb{N} \mid \ell(\lambda) \leqslant \ell \}$

· $SST(\lambda)$: the set of semistandard tableaux of shape $\lambda \in \mathcal{P}$

3.1. Spinor model

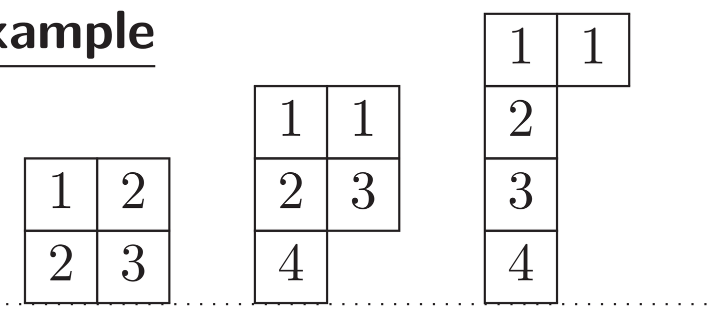
Spinor model [Kwon 15, Kwon 16] $(\lambda, \ell) \in \mathscr{P}(G)$

· $\mathbf{T}^{\mathfrak{g}}(a) = \bigsqcup_{n \geqslant 0} SST((2^n, 1^a))$ $(a \geqslant 0)$

· $\mathbf{T}^{\mathfrak{g}}(\lambda, \ell)$: subset of $\mathbf{T}^{\mathfrak{g}}(\lambda_\ell) \times \cdots \times \mathbf{T}^{\mathfrak{g}}(\lambda_1)$ satisfying the *admissibility condition*

· $\Pi^{\mathfrak{g}}(\lambda, \ell) = \Lambda^{\mathfrak{g}}_{\lambda_1} + \cdots + \Lambda^{\mathfrak{g}}_{\lambda_\ell} \in P^+$

Example



$(T_3, T_2, T_1) \in \mathbf{T}^{\mathfrak{g}}((3,1,0), 3)$

Thm [Kwon 15, Kwon 16] For $(\lambda, \ell) \in \mathscr{P}(G)$, $\mathbf{T}^{\mathfrak{g}}(\lambda, \ell)$ is a $\mathfrak{g}$-crystal and is isomorphic to $B(\Pi^{\mathfrak{g}}(\lambda, \ell))$, i.e.,

$$\mathbf{T}^{\mathfrak{g}}(\lambda, \ell) \cong B(\Pi^{\mathfrak{g}}(\lambda, \ell)).$$

3.2. $\mathfrak{g}_\infty$-type Kashiwara-Nakashima tableaux

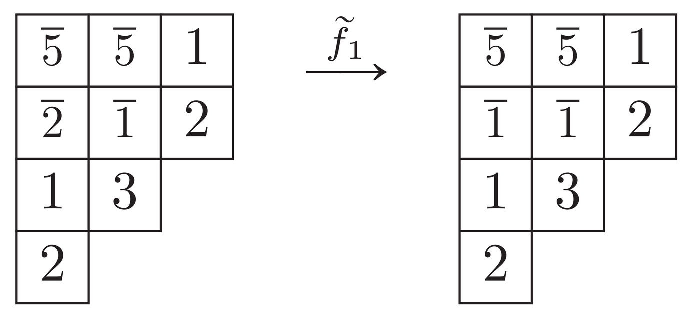
$\mathfrak{g}_\infty$-type KN tableaux [Lecouvey 09] $\lambda \in \mathcal{P}$

· $\mathcal{J}^{\mathfrak{g}} = \{ \cdots < \overline{3} < \overline{2} < \overline{1} < 1 < 2 < 3 < \cdots \}$

· $\mathbf{KN}^{\mathfrak{g}}(\lambda)$: subset of $SST(\lambda)$ whose letters are in $\mathcal{J}^{\mathfrak{g}}$ satisfying some *configuration conditions*

· $\varpi_\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_t\epsilon_t \in E$ $(t = \ell(\lambda))$

Example $\lambda = (3, 3, 2, 1)$



Thm [H. 23] For $\lambda \in \mathcal{P}$, $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ is a $\mathfrak{g}$-crystal and is isomorphic to $B(\varpi_\lambda)$, i.e.,

$$\mathbf{KN}^{\mathfrak{g}}(\lambda) \cong B(\varpi_\lambda).$$

4. The Grothendieck ring

· $\mathcal{K}$: Grothendieck ring of $\mathcal{C}$ (a category of extremal weight $\mathfrak{g}_\infty$-crystals), i.e., an additive group generated by isomorphism classes $[B]$ for $B \in \mathcal{C}$
Prop [Kwon 11, H. 23] $\mathcal{K}$ is an associative $\mathbb{Z}$-algebra.

· $\mathcal{K}^0$: the subalgebra of $\mathcal{K}$ generated by $[B(\lambda)]$ $(\lambda \in E)$
Prop [Lecouvey 09] There is an algebra isomorphism between $\mathcal{K}^0$ and the ring Sym of symmetric functions.

$$\Psi^0 : \mathcal{K}^0 \to \mathrm{Sym}, \quad [B(\varpi_\lambda)] \mapsto s_\lambda$$

· $\mathcal{K}^+$: subalgebra of $\mathcal{K}$ generated by $[B(\lambda)]$ $(\lambda \in P^+_{\mathrm{int}})$
Prop [H. 23] There is an algebra isomorphism between $\mathcal{K}^+$ and the ring $\mathbb{Z}[\![\mathbf{h}]\!]$ of formal power series in commuting variables $\mathbf{h} = \{\mathtt{h}_k \mid k \in \mathbb{Z}_+\}$.

$$\Psi^+ : \mathcal{K}^+ \to \mathbb{Z}[\![\mathbf{h}]\!], \quad [B(\Pi^{\mathfrak{g}}(\lambda, \ell))] \mapsto H^{\mathfrak{g}}(\lambda, \ell)$$

Remark $\mathcal{K} = \mathcal{K}^0 \otimes \mathcal{K}^+$ (as vector spaces)

· $\mathbf{z} = \{ \mathtt{z}_k \mid k \in \mathbb{N} \}$: commuting formal variables

· Let $\mathcal{A} = \sum_{n \geqslant 0} \mathcal{A}_n$, where $\mathcal{A}_0 = \mathbb{Z}[\![\mathbf{h}]\!]$, $\mathcal{A}_n = \mathcal{A}_0[\mathtt{z}_1, \ldots, \mathtt{z}_n]$ $(n \geqslant 1)$

⇝ $\mathcal{A}$ is an algebra under the following inductively-defined multiplication:
– $\mathcal{A}_0 = \mathbb{Z}[\![\mathbf{h}]\!]$: usual multiplication
– Suppose the multiplication on $\mathcal{A}_{n-1}$ is well-defined.
For $a \in \mathcal{A}_{n-1}$, define $az_n = z_n a + \delta_n(a)$ with a derivation δ_n on $\mathcal{A}_{n-1}$

$$\begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leqslant k \leqslant n-1) \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{\min\{a, n-i\}} \mathtt{z}_i \mathtt{h}_{a+n-i-2j} & (a \in \mathbb{Z}_+) \end{cases}$$

Thm [H. 23] There is an isomorphism of $\mathbb{Z}$-algebras.

$$\Psi : \mathcal{K} \to \mathcal{A}, \quad [B(\varpi_{(1^i)})] \mapsto \mathtt{z}_i, \ [B(\Pi_j^{\mathfrak{g}})] \mapsto \mathtt{h}_j$$

References

[1] T. Heo, *Extremal weight crystals over affine Lie algebras of infinite rank*, arXiv:2305.05188, 2023.

[2] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73.2** (1994), 383–-413.

[3] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994) 295–345.

[4] J.-H. Kwon, *Crystal duality and Littlewood-Richardson rule of extremal weight crystals*, J. Algebra **336** (2011), 99—138.

[5] J.-H. Kwon, *Super duality and crystal bases for quantum orthosymplectic superalgebras*, Int. Math. Res. Not. **23** (2015) 12620–12677.

[6] C. Lecouvey, *Crystal bases and combinatorics of infinite rank quantum groups*, Trans. Amer. Math. Soc. **361.1** (2009), 297–-329.

[7] S. Naito and D. Sagaki, *Tensor product multiplicities for crystal bases of extremal weight modules over quantum infinite rank affine algebras of types B_∞, C_∞, and D_∞"*, Trans. Amer. Math. Soc. **364.12** (2012), 6531—6564.

This research was supported by Basic Science Research Program through the NRF of Korea funded by the Ministry of Education (RS-2023-00241542).