Pop-Stack for Cambrian Lattices

(based upon arXiv:2312.03959)

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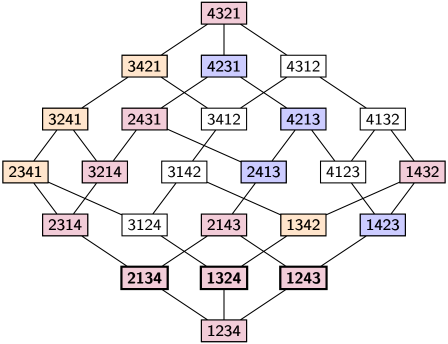
The weak order

- W a finite *Coxeter group*, S set of *simple reflections*.
- Example: $\mathfrak{S}_{n+1}$ the symmetric group.
  - Elements are bijections $w : [n+1] \to [n+1]$.
  - Simple $s_i = (i, i+1)$. These are the bold elements below.
- $\mathrm{Weak}(W)$ the right *weak order* on W.



$\mathrm{Weak}(\mathfrak{S}_4)$

(see the next panel for an explanation of the colors)

Cambrian lattices

- *Coxeter element* $c = s_{i_1} \cdots s_{i_n}$ product of all simple reflections (each appearing once) in some order.
- Camb_c the *Cambrian lattice* of c. This is the restriction of $\mathrm{Weak}(W)$ to the *c-sortable elements*.
- Example 1: $W = \mathfrak{S}_{n+1}$, $c^{\rightarrow} = s_1 \cdots s_n$.
  - Sortable elements are 312-avoiding.
  - $\mathrm{Camb}_{c^{\rightarrow}}$ is the Tamari lattice.
- Example 2: $W = \mathfrak{S}_{n+1}$, $c^{\times} = c^{\times}_{(n)} = \left(\prod_{i \text{ odd}} s_i\right)\left(\prod_{i \text{ even}} s_i\right)$.
  - Called the *bipartite* case.
  - Sortable elements avoid 312 with 2 odd and 231 with 2 even.
- On the previous panel:
  - Purple elements are both $c^{\rightarrow}$- and $c^{\times}$-sortable,.
  - Orange elements are $c^{\rightarrow}$-sortable but not $c^{\times}$-sortable.
  - Blue elements are $c^{\times}$-sortable but not $c^{\rightarrow}$-sortable.

Pop-stack operators

Let L be a lattice with meet operation $\wedge$ and join operation $\vee$. The *pop-stack operator* $\mathrm{pop}_L^{\downarrow}\colon L \to L$ is defined by

$$\mathrm{pop}_L^{\downarrow}(x) = x \wedge \left(\bigwedge \{ y \mid y \lessdot x \} \right),$$

where we write $u \lessdot v$ to mean that u is covered by v in L. The *dual pop-stack operator* $\mathrm{pop}_L^{\uparrow} : L \to L$ is defined analogously.

Historical Example [Ung82]: For $L = \mathrm{Weak}(\mathfrak{S}_{n+1})$, $\mathrm{pop}_L^{\downarrow}$ reverses descending runs[1]:

$$3241 \xrightarrow{\mathrm{pop}^{\downarrow}} 2314 \xrightarrow{\mathrm{pop}^{\downarrow}} 2134 \xrightarrow{\mathrm{pop}^{\downarrow}} 1234$$

Example [Def22]: For any W, $\mathrm{pop}_{\mathrm{Weak}(W)}^{\downarrow}(w) = w \cdot w_\circ(\mathrm{Des}(w))$ (multiply by the longest word in the right descent set of w).

[1] This is the original definition of $\mathrm{pop}^{\downarrow}$ for permutations. It is equivalent to the version defined here (with $L = \mathrm{Weak}(\mathfrak{S}_{n+1})$) by [Def22].

Canonical join complex

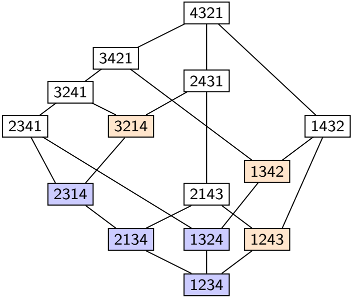
- Camb_c is *semidistributive*, so every element $w \in \mathrm{Camb}_c$ has a *canonical join representation* $\mathcal{D}(w)$ and a *canonical meet representation* $\mathcal{U}(w)$.
- Explicitly, $\mathcal{D}(w)$ is the antichain joining to w which generates the smallest possible order ideal. $\mathcal{U}(w)$ is constructed dually.
- The collection of canonical join (resp. meet) representations is a flag simplicial complex. The *canonical join complex* and *canonical meet complex* are isomorphic [Bar19].
- An element $x \in \mathrm{Camb}_c$ is in the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$ (resp. $\mathrm{pop}^{\uparrow}_{\mathrm{Camb}_c}$) if and only if its canonical meet (resp. join) representation is a facet of the canonical meet (resp. join) complex [DW23].

Examples of the image of $\mathrm{pop}^{\downarrow}$

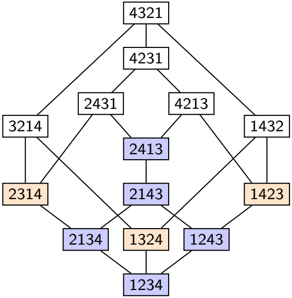
Example [ABB+19]: $w \in \mathfrak{S}_{n+1}$ is in the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(\mathfrak{S}_{n+1})}$ if and only if each adjacent pair of ascending runs is overlapping.

Example [Hon22]: $w \in \mathrm{Camb}_{c^{\rightarrow}}$ is in the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\rightarrow}}}$ if and only if w contains no *double descents* (i.e., there are no descending runs of length > 2) and $w(n+1) = (n+1)$.

Examples: elements of the image are shaded blue.



$\mathrm{Camb}_{c^{\rightarrow}}$



$\mathrm{Camb}_{c^{\times}}$

Characterization of the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$

Now let W and c be arbitrary. For $s_i \in S$, denote

$$p_i = \bigvee \{w \in \mathrm{Camb}_c \mid s_i \leq w \text{ and } s_j \not\leq w \text{ for all } s_j \in S \setminus \{s_i\}\}.$$

(These elements are shaded orange in the examples.)

Theorem (Barnard-Defant-Hanson [BDH])

Let $w \in \mathrm{Camb}_c$. Then the following are equivalent.

1. *w is in the image of $\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}$.*
2. *the right descents of w all commute and w has no left inversions in common with c^{-1}.*
3. *The interval $[\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w), w]$ is Boolean and $p_i \not\leq w$ for all $s_i \in S$.*

Quiver representations lurking in the background!

We prove the theorem above using *representations of quivers*:

- The Coxeter element c induces an orientation Q of the *Coxeter graph* of W.
- For W crystallographic[2], the lattice Camb_c can be modeled as the *lattice of torsion classes* of representations of Q [IT09].
- The canonical join representation $\mathcal{D}(w)$ of some $w \in \mathrm{Camb}_c$ can be encoded as a set of representations $\mathcal{X}(w)$ [BCZ19].
- The condition on descents commuting translates to there being no *extensions* between the representations in $\mathcal{X}(w)$.
- The condition on inversions (and on the elements p_i) translates to there being no *projective representations* in $\mathcal{X}(w)$.

[2] The non-crystallographic cases are proved by direct computation.

Iterations of pop-stack

The following is a consequence of the first theorem and additional representation-theoretic arguments:

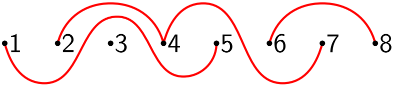
Theorem (Barnard-Defant-Hanson [BDH])

Let $w \in \mathrm{Camb}_c$. *Then, for* $t \geq 0$,

$$\left(\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}\right)^{t+1}(w) = \left(\mathrm{pop}^{\downarrow}_{\mathrm{Weak}(W)}\right)^{t}(\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_c}(w)).$$

Arc diagrams

- For $w \in \mathfrak{S}_{n+1}$, one obtains a *noncrossing arc diagram* $\Delta(w)$ by graphing the descending runs of w. For example:



$$w = 86|742|51|3 \in \mathfrak{S}_8$$

where e.g. the arc 42 passes over 3 because $w^{-1}(4) < w^{-1}(3)$.

- We denote $\mathrm{AD}(c^\times) = \{\Delta(w) \mid w \in \mathrm{Camb}_{c^\times}\}$. By [Rea15], an arc diagram δ is in $\mathrm{AD}(c^\times)$ if and only if no arc of δ passes above an even node or below an odd node.

- The arcs in $\Delta(w)$ correspond to the canonical joinands of w, so the facets of the canonical join complex correspond to the set $\mathrm{MAD}(c^\times)$ of maximal elements of $\mathrm{AD}(c^\times)$ with respect to inclusion of sets of arcs.

Motzkin paths

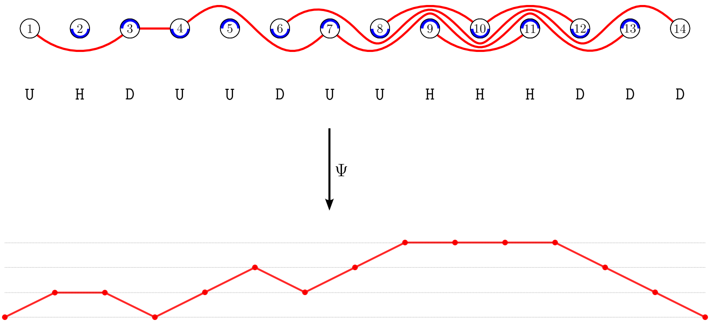
- A *Motzkin path* is a lattice path in the plane that consists of up ($\mathtt{U} = (1,1)$) steps, down ($\mathtt{D} = (1,-1)$) steps, and horizontal ($\mathtt{H} = (1,0)$) steps, starts at the origin, never passes below the horizontal axis, and ends on the horizontal axis. (An example is given below.)
- Let $\overline{\mathcal{M}}_n$ be the set of Motzkin paths of length n that have no *peaks of height* 1 (i.e., that do not pass through all of the points $(i,0),(i+1,1)$, and $(i+2,0)$ for any $i \in \mathbb{N}$).
- Suppose $\delta \in \mathrm{MAD}(c^\times)$. Let $\Psi(\delta)$ be the word $\mathtt{M}_1 \cdots \mathtt{M}_{n+1}$, where for $1 \leq i \leq n+1$, we define

$$\mathtt{M}_i = \begin{cases} \mathtt{U} & \text{if } i \leq n \text{ and } i+1 \text{ is not the right endpoint of an arc in } \delta; \\ \mathtt{D} & \text{if } i \geq 2 \text{ and } i-1 \text{ is not the left endpoint of an arc in } \delta; \\ \mathtt{H} & \text{otherwise.} \end{cases}$$

Bijection

Theorem (Barnard-Defant-Hanson [BDH])

1. *The map* Ψ *is a bijection from* $\mathrm{MAD}(c^{\times})$ *to* $\overline{\mathcal{M}}_{n+1}$.
2. *For each* $\delta \in \mathrm{MAD}(c^{\times})$, *we have* $|\delta| = n - \#_{\mathtt{U}}(\Psi(\delta))$.



Enumeration

Let $I^{\downarrow} = \mathrm{image}\left(\mathrm{pop}^{\downarrow}_{\mathrm{Camb}_{c^{\times}_{(n)}}}\right)$, $I^{\uparrow} = \mathrm{image}\left(\mathrm{pop}^{\uparrow}_{\mathrm{Camb}_{c^{\times}_{(n)}}}\right)$, and

$$\mathbf{P}_{\mathrm{Camb}_{c^{\times}_{(n)}}}(q) := \sum_{w \in I^{\downarrow}} q^{|\mathcal{U}(w)|} = \sum_{w \in I^{\uparrow}} q^{|\mathcal{D}(w)|} = \sum_{\delta \in \mathrm{MAD}(c^{\times})} q^{|\delta|}.$$

An enumerative consequence of the bijection Ψ is:

$$\sum_{n \geq 1} \mathbf{P}_{\mathrm{Camb}_{c^{\times}_{(n)}}}(q) z^n = \frac{1}{qz}\left(\frac{2}{1 - qz(1-2z) + \sqrt{1 + q^2 z^2 - 2qz(1+2z)}} - 1\right) - 1.$$

(See OEIS A089372.)

Quotients of the weak order

- The *Coxeter number* of W is the quantity $h = 2|T|/|S|$, where T is the set of reflections in W.
- For L a lattice and $x \in L$, denote

$$\mathcal{O}_L(x) = \{x, \mathrm{pop}_L^{\downarrow}(x), (\mathrm{pop}_L^{\downarrow})^2(x), \ldots\}.$$

- By [Def22] (and [Ung82] in type A), $\max_{x \in W} \left|\mathcal{O}_{\mathrm{Weak}(W)}(x)\right| = h$.

Theorem (Barnard-Defant-Hanson [BDH])

If $W_{\equiv}$ is a lattice quotient of $\mathrm{Weak}(W)$, then $\max_{x \in W_{\equiv}} |\mathcal{O}_{W_{\equiv}}(x)| \leq h$.

Theorem (Barnard-Defant-Hanson [BDH])

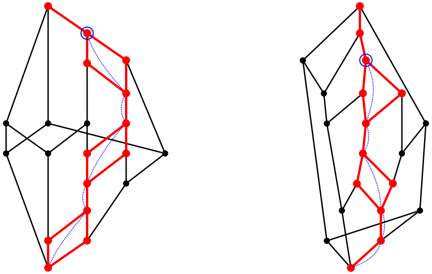
For each Coxeter element c of W, we have $\max_{x \in W_{\equiv}} \{|\mathcal{O}_{\mathrm{Camb}_c}(x)|\} = h$.

Maximizers

Elements $\mathbf{z}_c$ realizing $|\mathcal{O}_{\mathrm{Camb}_c}(\mathbf{z}_c)| = h$ are obtained by applying $\mathrm{pop}^{\uparrow}$ in the *spine*[3] (the union of maximal length chains) of Camb_c. Precisely,

$$\mathbf{z}_c = \left(\mathrm{pop}^{\uparrow}_{\mathrm{spine}(\mathrm{Camb}_c)}\right)^{h-1}(e),$$

where $e = \hat{0}$ is the identity element. For example (in type B_3):



[3] This is a distributive sublattice of Camb_c by [HLT11].

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