Determinant of the distance matrix of a tree

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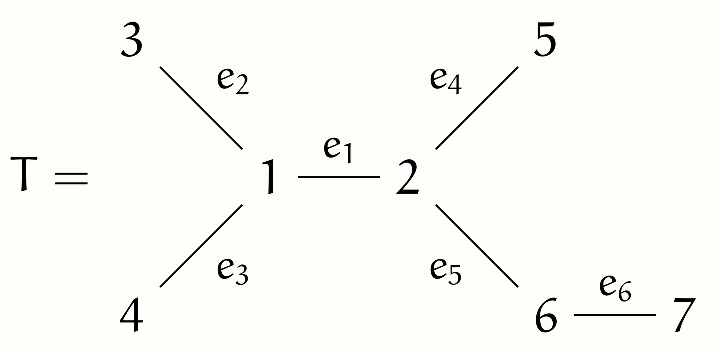
Graham and Pollak's Theorem

Theorem (Graham–Pollak, 1971). *The determinant of the distance matrix $M(T)$ of a tree T depends only on its number of vertices n. Precisely:*

$$\det M(T) = (-1)^{n-1}(n-1)2^{n-2}.$$



$$\det M(T) = \det \begin{pmatrix} 0&2&1&2&3&3&4\\ 2&0&1&2&3&3&4\\ 1&1&0&1&2&2&3\\ 2&2&1&0&1&1&2\\ 3&3&2&1&0&2&3\\ 3&3&2&1&2&0&1\\ 4&4&3&2&3&1&0 \end{pmatrix} = +6\cdot 2^5.$$

Elementary proofs are known, but...

What does $(n-1)2^{n-2}$ count?

We give the first *combinatorial proof*.

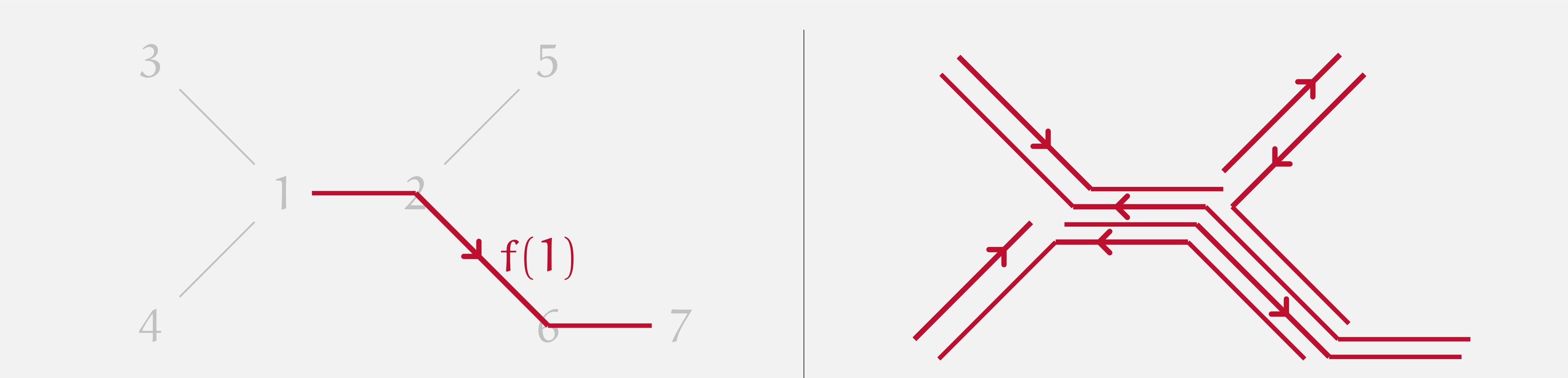
Catalysts and arrowflows

$$\det M(T) = \sum_{\sigma\in\mathbb{S}_n} \epsilon(\sigma)\prod_i d(i,\sigma(i))$$

$\prod_i d(i,\sigma(i))$: # edges between i and $\sigma(i)$

pairs (σ, f) with $\sigma\in\mathbb{S}_n$, $f : V \to E$ such that $f(i)$ is an edge between i and $\sigma(i)$ for all i

We call such a pair a *catalyst*; we have $\det M(T) = \sum_{(\sigma,f)\text{ catalyst}} \epsilon(\sigma)$.



A marked path $(P(1,7); e_5)$ from 1 to 7 marked at $e_5 = \{2,6\}$.

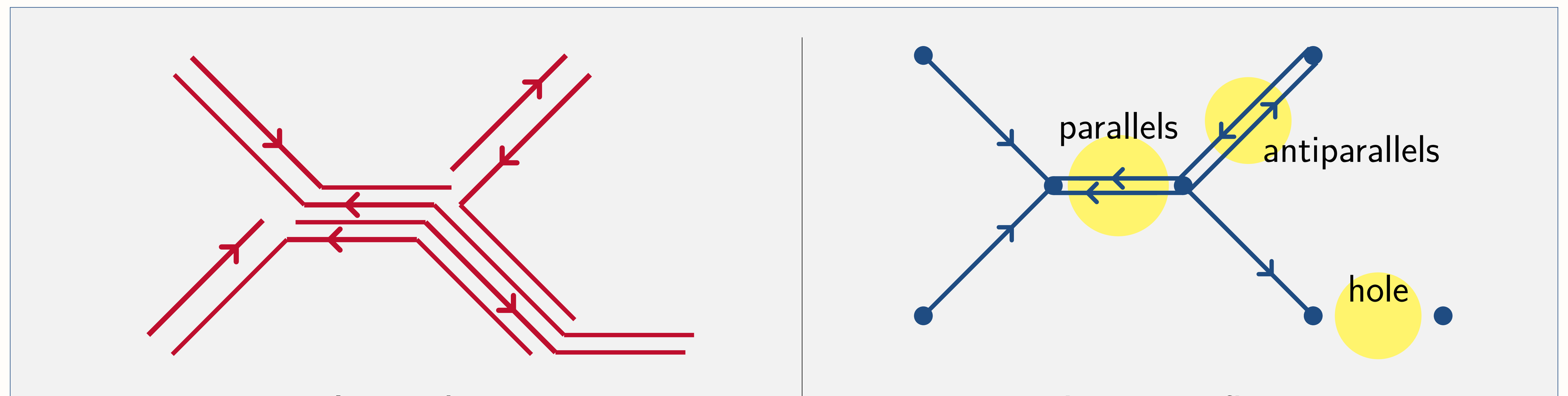
A catalyst (σ, f), where $\sigma = 7521643$ and $f = e_5e_4e_2e_3e_4e_1e_1$.

Goal. Define involutions on the set of catalysts such that

$$\det M(T) = \pm\ \#\text{fixed points of the involution}.$$

Arrowflows

Definition. An *arrowflow* is a multiset of n directed edges of $\vec{E}$.



A catalyst. Its arrowflow.

We have $\det M(T) = \sum_{A\text{ arrowflow}} \sum_{(\sigma,f)\text{ catalyst of }A} \epsilon(\sigma)$.

Combinatorial proof of Graham and Pollak's Theorem

Theorem. $\sum_{(\sigma,f)\text{ catalyst of }A} \epsilon(\sigma) = \begin{cases} 0 & \textit{if } A \textit{ has holes or parallels,}\\ (-1)^{n-1} & \textit{otherwise.}\end{cases}$

An arrowflow without holes or parallels has exactly one antiparallel pair.

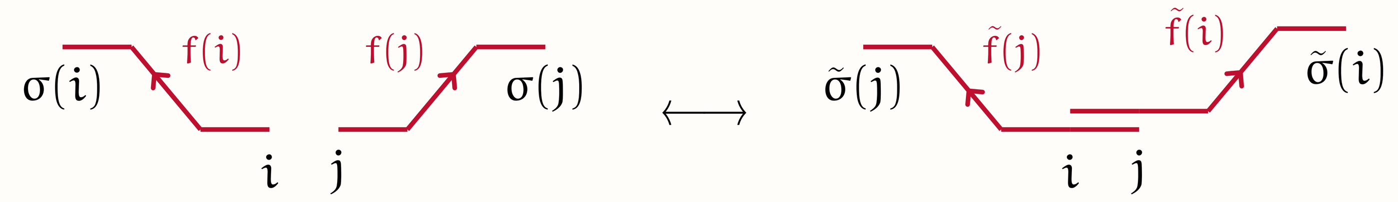
Corollary. $\det M(T) = (-1)^{n-1}(n-1)\,2^{n-2}$.

$(n-1)$: choices for the antiparallel pair

2^{n-2}: choices for the orientations of the single arcs

Proof of theorem (sketch).

1. Sign-reversing involution to kill catalysts whose arrowflow has holes or parallels.



2. For every other A, construction of a network $\mathcal{R}_A$ (the *route map*) such that

$$\{\text{catalysts of } A\} \overset{1:1}{\longleftrightarrow} \{n\text{-families of paths in } \mathcal{R}_A\}$$

and such that there is a unique non-intersecting family in $\mathcal{R}_A$.

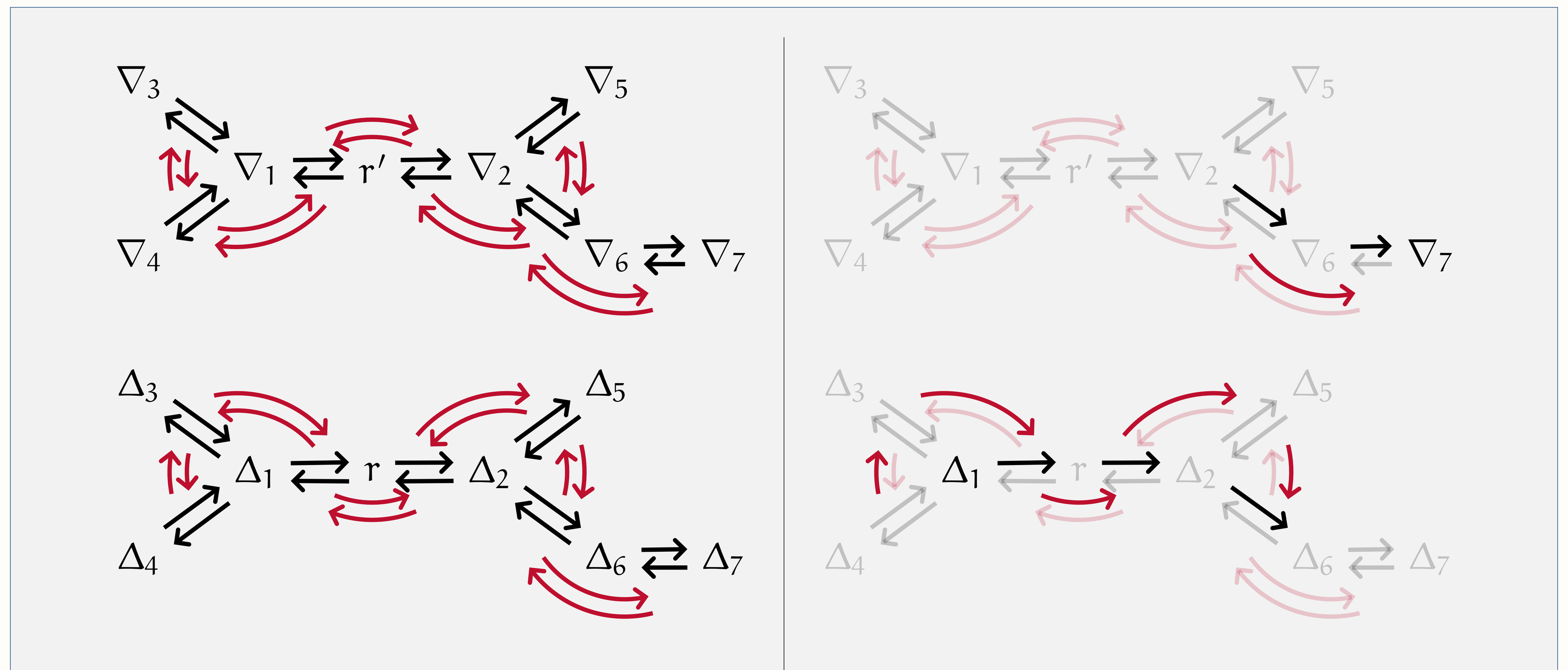
3. **Lindström–Gessel–Viennot Lemma.** *Given a network $\mathcal{R}$ with sources $\{\Delta_1, \dots, \Delta_n\}$ and sinks $\{\nabla_1, \dots, \nabla_n\}$,*

$$\sum_{P=(P_1,\dots,P_n)} \epsilon(\sigma_P) = \sum_{\substack{P=(P_1,\dots,P_n)\\ \textit{non-intersecting}}} \epsilon(\sigma_P)$$

where P_i is a path $\Delta_i \to \nabla_{\sigma_P(i)}$.

Route maps

The route map $\mathcal{R}_A$ is the union of two *hemisphere* networks, $\mathcal{S}$ and $\mathcal{N}$. Nodes of $\mathcal{S}$ and $\mathcal{N}$ correspond to *vertices*, *oriented edges* and *oriented sectors* of a planar embedding of a rooted subdivision of T. Paths in $\mathcal{R}_A$ lift marked paths in T.



Nodes of $\mathcal{R}_A$. Lift of the marked path $(P(1,7); e_5)$.

Generalizations

Introduce variables x_{ij}, x_{ji}, y_{ij}, y_{ji}, z_{ij}, z_{ji} for $\{i,j\}\in E$. Set $x_{ji} = x_{ij}^{-1}$ for all i, j.

Let $w(P(1,7); e_5) = x_{12}y_{26}z_{67}$, let $w(\sigma, f) = \prod_i w(P(i,\sigma(i)); f(i))$.

Let $M'(T)_{ij} = \sum_{e\in P(i,j)} w(P(i,j); e)$.

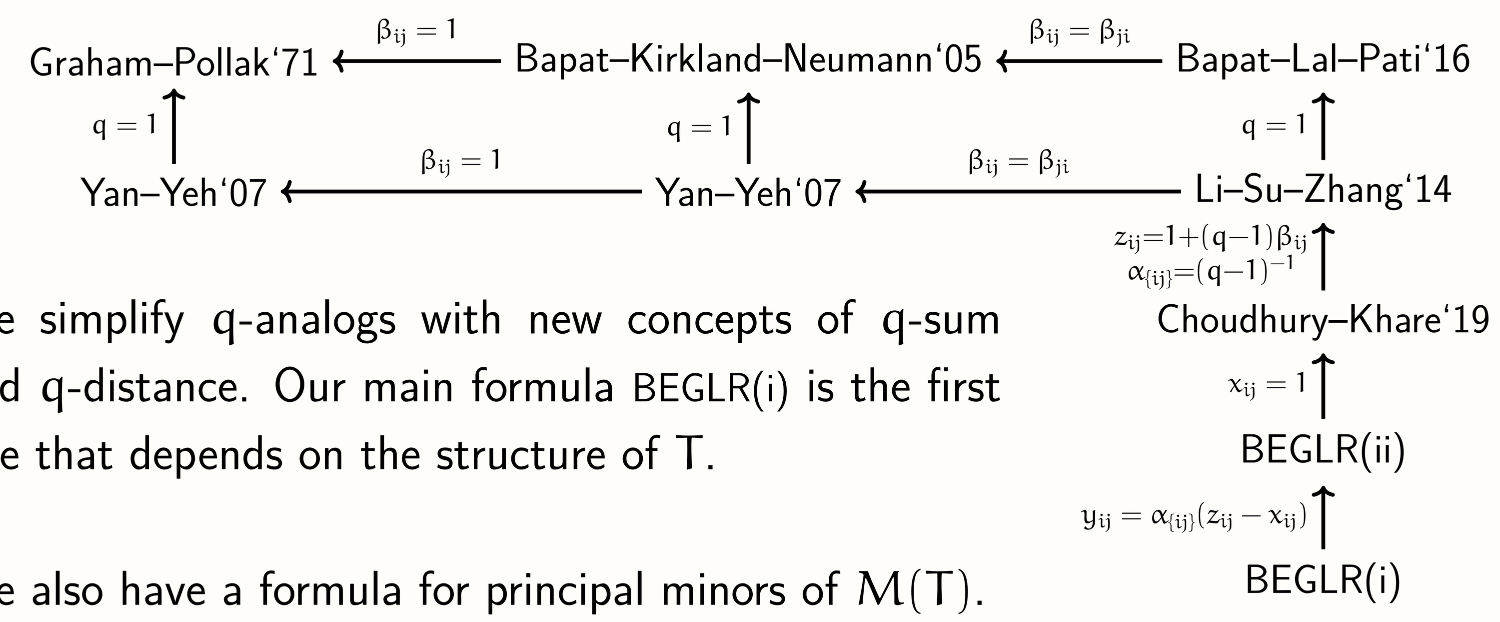
Theorem.

$$\det M'(T) = (-1)^{n-1}\sum_{\{a,b\}\in E} y_{ab}y_{ba}\prod_{(i,j)}(y_{ij}x_{ji} + y_{ji}z_{ij}),$$

as (i,j) ranges over oriented edges "pointing to $\{a,b\}$".

Proof. All of the involutions are weight-preserving. ■

This implies *every generalization* in the literature, and two new ones.



We simplify q-analogs with new concepts of q-sum and q-distance. Our main formula BEGLR(i) is the first one that depends on the structure of T.

We also have a formula for principal minors of $M(T)$.

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