The somewhere-to-below shuffles in the symmetric group and Hecke algebras

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Summary

- We define the **somewhere-to-below shuffles**, a family of elements of the symmetric group algebra.
- These elements are **simultaneously triangularizable** in a **combinatorial basis** of the symmetric group algebra that we introduce.
- We give a **filtration of $\mathbf{k}[S_n]$-modules** invariant under the somewhere-to-below shuffles.
- The **eigenvalues** of the somewhere-to-below shuffles and their linear combinations have an easy expression.
- We interpret the somewhere-to-below shuffles and their linear combination as **card shuffling** methods, and give a strong stationary time for the random-to-below shuffle.

Somewhere-to-below shuffles

- Cycles: $\operatorname{cyc}_{i_1,i_2,\ldots,i_m}$ denotes the m-cycle sending $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_m \mapsto i_1$
- We look at cycles made of adjacent cards: $\operatorname{cyc}_{i,i+1,\ldots,j}$ denotes the cycle sending $i \mapsto i+1 \mapsto \cdots \mapsto j \mapsto i$
- The n somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$ are

$$t_i = \operatorname{cyc}_i + \operatorname{cyc}_{i,i+1} + \ldots + \operatorname{cyc}_{i,i+1,\ldots,n}.$$

- They are called *somewhere-to-below* due to their card shuffling interpretation: t_i represents the act of taking the card at position i and moving it anywhere weakly below.

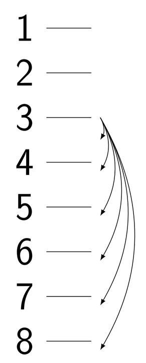

Figure 1: $t_3 \in S_8$

Right action

We view the elements of the symmetric group as *endomorphisms* of the symmetric group algebra. They act on the right (action on positions, aka right multiplication).

Linear combinations

The linear combinations of somewhere-to-below shuffles are called *one-sided cycle shuffles*. Two of them are of particular interest.

Top-to-random

Top-to-random corresponds to $\frac{1}{n}t_1$, and has been well-studied as a card shuffling technique. Its eigenvalues are known to be $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1\}$.

Random-to-below

Random-to-below is the shuffle that picks a card uniformly at random, and inserts it weakly below, uniformly at random. This is $\frac{1}{n^2}t_1 + \frac{1}{n(n-1)}t_2 + \frac{1}{n(n-2)}t_3 + \cdots + \frac{1}{n}t_n$.

Lacunar sets

Set $[k] := \{1, 2, \ldots, k\}$ for any k.
A *lacunar set* is a subset of $[n-1]$ that does not contain consecutive integers.

Known fact: The number of lacunar sets of $\{1, 2, \ldots, n-1\}$ is the $(n+1)$-th Fibonacci number.

For a lacunar set I and an integer $\ell \in [n]$, we define the number

$$m_{I,\ell} = (\text{smallest element of } I \cup \{n+1\} \text{ that is } \geq \ell) - \ell \in [0, n].$$

Example

For $n = 4$, the following five subsets of $\{1, 2, 3\}$ are lacunar:

I	$m_{I,1}$	$m_{I,2}$	$m_{I,3}$	$m_{I,4}$
$\emptyset$	4	3	2	1
$\{1\}$	0	3	2	1
$\{2\}$	1	0	2	1
$\{3\}$	2	1	0	1
$\{1,3\}$	0	1	0	1

Descent-destroying basis

Definition

For each $w \in S_n$, we define the *descent set* of w to be the set

$$\operatorname{Des} w := \{i \in [n-1] \mid w(i) > w(i+1)\}.$$

For each $i \in [n-1]$, we define the *simple transposition* $s_i := \operatorname{cyc}_{i,i+1} \in S_n$.
For each $w \in S_n$, we define

$$a_w := \sum_{\sigma \in \langle s_i \mid i \in \operatorname{Des} w \rangle} w\sigma \in \mathbf{k}[S_n].$$

Proposition

The family $(a_w)_{w \in S_n}$ is a basis of $\mathbf{k}[S_n]$. We call it the *descent-destroying basis*.

Example

For $n = 3$, we have

$$a_{[123]} = [123]; \qquad a_{[231]} = [231] + [213];$$

$$a_{[132]} = [132] + [123]; \qquad a_{[312]} = [312] + [132];$$

$$a_{[213]} = [213] + [123]; \qquad a_{[321]} = [321] + [312] + [231] + [213] + [132] + [123]$$

Theorem

The endomorphisms representing the right action by each of $t_1, t_2, \ldots, t_n$ are upper triangular in the descent-destroying basis. That is: $a_w t_i = \sum_{u \leq w} \lambda_u a_u$ for some total order $\leq$ on S_n.

Fibonacci filtration

We define the following invariant spaces, defined for each lacunar set $I \subseteq [n-1]$:

$$F(I) := \{q \in \mathbf{k}[S_n] \mid qs_i = q \text{ if } i \notin I \text{ and } i+1 \notin I\}.$$

We set a total order on the lacunar sets with the constraint that $\operatorname{sum}(I) \leq \operatorname{sum}(J)$ whenever $I \leq J$. We index the lacunar sets:

$$Q_1 < Q_2 < \ldots < Q_{f_{n+1}}.$$

Let

$$F_i := \sum_{j \leq i} F(Q_j).$$

Theorem

The right action by any of the somewhere-to-below shuffles preserves each module F_i, and the shuffle t_ℓ acts on the quotients F_i/F_{i-1} as multiplication by the scalar $m_{Q_i,\ell}$.

Eigenvalues

Theorem

Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the eigenvalues of the endomorphism acting on the right by $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ are the linear combinations

$$\lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar}$$

(with known multiplicities). If all these f_{n+1} linear combinations are distinct, then the endomorphism is diagonalizable.

Card shuffling

Linear combinations of somewhere to-below shuffles have a probabilistic meaning, when the coefficients λ_i are nonnegative, as they model the following shuffle.

- Pick the i-th card from the top with probability to $\frac{1}{n-i+1} \cdot \frac{\lambda_i}{\sum_{k=1}^{n} \lambda_k}$.
- Insert it at a position below chosen uniformly at random.

Mixing times: How many times should we shuffle?

Theorem

If $\lambda_1 \neq 0$, then the one-sided cycle shuffle $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ admits a stopping time τ obtained as follows:

- Place a bookmark right above the bottommost card of the deck.
- The bookmark itself does not move down (but cards can move down past it).
- We let τ be the time it takes for the bookmark to reach the top of the deck.

The distribution of the deck is uniform at time τ and any time afterwards; so τ is a strong stationary time. This stopping time is optimal.
The expected number of steps to get to the strong stationary time for the random-to-below shuffle is

$$\mathbb{E}(\tau) = \sum_{i=2}^{n} \frac{n}{i(H_n - H_{i-1})} \leq n \log n + n \log(\log n) + n \log 2 + 1 \qquad \text{if } n \geq 2,$$

where H_n is the n-th harmonic number $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

- It means that random-to-below is slower than top-to-random.
- We conjecture that the bound on $\mathbb{E}(\tau)$ is optimal for the random-to-below shuffle.

Nilpotent commutators

- The pairwise commutators $[t_i, t_j]$ in $\mathbf{k}[S_n]$ are nilpotent (by the upper-triangularity above).
- How small is the required exponent? Much smaller than one might expect!

Theorem

Let $1 \leq i \leq j \leq n$. Then, $[t_i, t_j]^m = 0$ holds for $m = \min\{j - i + 1, \lceil (n-j)/2 \rceil + 1\}$.

We conjecture (and have verified for all $n \leq 12$) that this choice of m is optimal (for $\mathbf{k} = \mathbb{Z}$).

More

We have ...

- combinatorial formulas for the multiplicities of the eigenvalues;
- a sufficient condition for diagonalizability;
- eigenvalues on each Specht module;
- a conjectural q-deformation.

References

- Darij Grinberg and Nadia Lafrenière. *The one-sided cycle shuffles in the symmetric group algebra*, Algebraic Combinatorics, volume 7, issue 2, p. 275-326 2024.
- Darij Grinberg. *Commutator nilpotency for somewhere-to-below shuffles*. ArXiv:2309.05340v2, 2023.