Antichains in the representation theory of finite lattices

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Notation: Let L be a finite lattice. - For all $x \in L$, let P_x, S_x, be the projective indecomposable, simple, module associated to x. - When $\mathcal{T} = D^b(A)$, the bounded derived category of an algebra A of finite global dimension over a field $\Bbbk$, the Serre functor is $\mathbb{S} = - \otimes^{\mathbb{L}}_{A} DA$, the derived Nakayama functor. - Let $[d]$ denote the suspension functor applied d times.
Layout of this poster: context | results | toolbox

The results of the rightmost column of this poster rely on the following isomorphism between the total hom complex from a perfect complex to a module.

$$\begin{array}{ccccccc} \dots \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(\bigoplus_{x\in S_n} P_x, M) & \xrightarrow{\partial^*_{n+1}} & \operatorname{Hom}_{\mathcal{A}}(\bigoplus_{x\in S_{n+1}} P_x, M) & \longrightarrow \dots \\ & \downarrow & & \downarrow & \\ \dots \longrightarrow & \bigoplus_{x\in S_n} e_x M & \longrightarrow & \bigoplus_{x\in S_{n+1}} e_x M & \longrightarrow \dots \end{array}$$

Context

Calabi-Yau objects first appeared in geometry but have come a long way since. A triangulated $\Bbbk$-linear category $\mathcal{T}$ with a Serre functor $\mathbb{S}$ is said to be *fractionally Calabi-Yau* if there exist l, d such that $\mathbb{S}^l$ is isomorphic as a functor to $[d]$. Applying this notion to incidence algebras of posets, a conjecture by Chapoton links combinatorial formulas, fractionally Calabi-Yau posets and symplectic geometry.

Chapoton's conjecture [5]

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of non negative integers. Assume that for all integer n there exist integers $m, D, d_1, \dots, d_m$ satisfying the *product formula*

$$s_n = \prod_{i=1}^{m} \frac{D - d_i}{d_i}.$$

Then there should exist a family of posets $(P_n)_{n\in N}$ with cardinals $|P_n| = s_n$ such that the bounded derived category $D^b(P_n)$ is fractionally Calabi-Yau of dimension $\frac{C}{D}$ where

$$C = \sum_{i=1}^{m} D - 2d_i.$$

Consider a quasi homogenous singularity f of degrees $d_1, \dots, d_m$ and total degree D. Then there should exist a geometric triangulated (Fukaya-Seidel) category $\mathcal{F}_n$ associated to f which is equivalent to $D^b(P_n)$

Product formula with $D = m + n + 1$

The Calabi–Yau dimension matches the prediction

Examples

Theorem 1 ([3, Theorem 8.3]). *Let $n \in \mathbb{N}$. The bounded derived category $D^b(Tam_n)$ is $\frac{n(n-1)}{2n+2}$ fractionally Calabi-Yau.*

Theorem 2 ([2, Theorem 4.1]). *The Serre functor has finite order on the Grothendieck group of the incidence algebra $\mathcal{A}$ of the poset of order ideals $J(P_{m,n})$ of a grid poset $P_{m,n}$.*

References

[1] Osamu Iyama. "Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories." In: *Adv. Math.* 210.1 (2007), pp. 22–50. ISSN: 0001-8708. DOI: 10.1016/j.aim.2006.06.002.

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[3] Baptiste Rognerud. "The bounded derived categories of the Tamari lattices are fractionally Calabi-Yau". In: *Adv. Math.* 389 (2021), p. 31. ISSN: 0001-8708. DOI: 10.1016/j.aim.2021.107885.

[4] Osamu Iyama and René Marczinzik. "Distributive lattices and Auslander regular algebras". In: *Adv. Math.* 398 (2022). Id/No 108233, p. 27. ISSN: 0001-8708. DOI: 10.1016/j.aim.2022.108233.

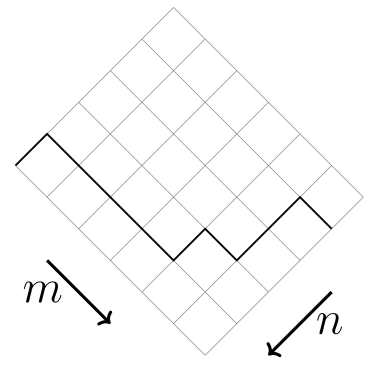
[5] Frédéric Chapoton. *Posets and Fractional Calabi-Yau Categories.* 2023. arXiv: 2303.11656 [math.CO].

[6] Ilaria Di Dedda. *Symplectic higher Auslander correspondence for type A.* 2024. arXiv: 2311.16859.

Order ideals of the product of two chains

We denote this *lattice* $J_{m,n}$ and there is an isomorphism of lattices $J_{m,n} \overset{\sim}{\longleftrightarrow} \{(a_1, \dots, a_m) | \text{non decreasing in} [\![0, n]\!]\}$.

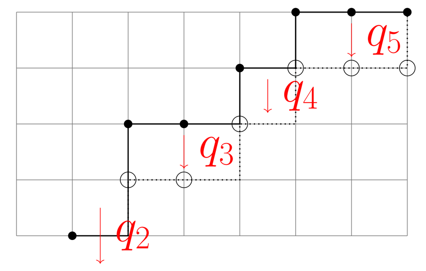
$$\Rightarrow |J_{m,n}| = \binom{m+n}{m} = \frac{m+n}{1} \times \dots \times \frac{n+1}{m}.$$



Antichains in $J_{m,n}$

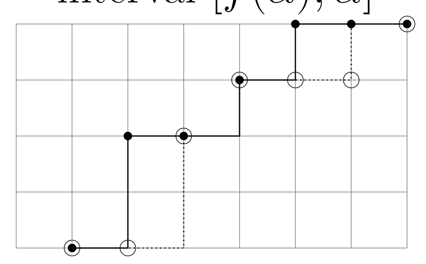
For each $\alpha \in J_{m,n}$ we construct an antichain C_α. Denote $\mathcal{P}_\alpha$ its object in $D^b(J_{m,n})$.

$C_\alpha = \{q_i(\alpha) | i \in S_\alpha\}$



$\rightsquigarrow$ The associated antichain module is an interval with minimum $f(\alpha)$[2, Proposition 2.13] $\rightsquigarrow$

The bounds of the interval $[f(\alpha), \alpha]$



$D^b(J_{m,n})$ is fractionally Calabi–Yau

Proposition 1. $\mathbb{S}^{m+n+1}(\mathcal{P}_\alpha) \cong \mathcal{P}_\alpha[mn]$.

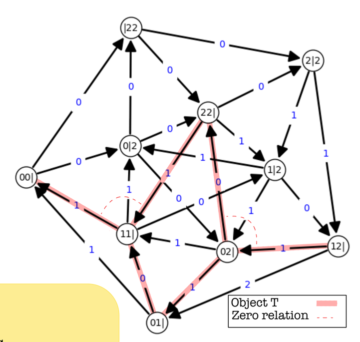
Proposition 2. *The antichain C_α is boolean and in particular strong*

$\Rightarrow D^b(J_{m,n})$ is $\frac{mn}{m+n+1}$-Calabi–Yau.

Theorem 3 applies

Change of basis

We give a presentation by generators and relations for the category spanned by the objects $\mathcal{P}_\alpha$ and their shifts. By doing a sort of *change of basis* in the derived category we find an algebra which is derived equivalent to $J_{m,n}$: the Higher Auslander Algebra of type A^{n-1}_{m+1}. It is a well known object in higher representation theory[1].



Theorem 4 applies

Towards a geometric model

Higher Auslander Algebras of type A have a geometric model which seems to fit Chapoton's prediction[6]. This gives a first example which corroborates all three aspects of this surprising conjecture.

From antichains to complexes

Definition 1. An antichain C of L is a set of pairwise incomparable elements of L.

Core idea (*e.g.*[4, Proposition 2.1]):

$$\{\text{antichains}\} \longleftrightarrow \{\text{submodules of } P_{\hat{1}}\} \longleftrightarrow \{\text{modules with head } S_{\hat{1}}\}$$

$$C \longmapsto N_C := \sum_{i=1}^{r} A \cdot (c_i, \hat{1}) \longmapsto M_C = P_{\hat{1}}/N_C \quad (1)$$

M^{α}_C is called an *antichain module*. Let $\mathcal{P}^{\alpha}_C$ be its *antichain projective resolution*

$$0 \to P_r \to \dots \to P_0 \to M_C \text{ where } P_0 = P_{\hat{1}} \text{ and } P_l = \bigoplus_{\substack{S\subseteq C\\|S|=l}} P_{\wedge S} \text{ for } 1 \le l \le r. \quad (2)$$

With boundary maps defined by

$$\begin{array}{ccc} P_{\wedge S} & \to & P_{\wedge T} \\ (x, \wedge S) & \mapsto & \begin{cases} (-1)^{|i|_S}(x, \wedge T) & \text{if } T \sqcup \{i\} = S, \\ 0 & \text{otherwise} \end{cases} \end{array} \quad (3)$$

Note: Intervals are antichain modules

Rigidity properties

Inclusive antichain For all subsets S and S' of C, if $\wedge S \le \wedge S'$ then $S' \subseteq S$

Intersective antichain For all subsets S and S' of C, we have $(\wedge S) \vee (\wedge S') = \wedge(S \cap S')$.

Strong antichain For all S, S' subsets of C of same cardinal, $\wedge S$ and $\wedge S'$ are incomparable *i.e* if $\wedge S \le \wedge S'$ then $S = S'$.

Boolean antichain C is both inclusive and intersective.

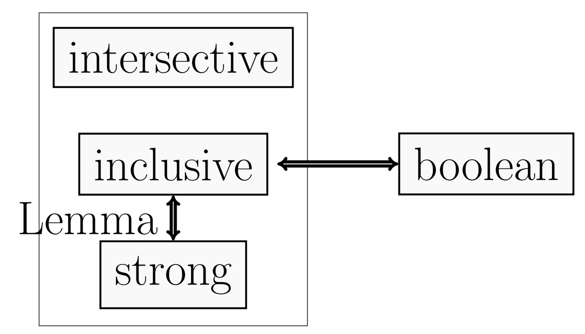


Fig. 3: Properties of antichains

About the terminology: We can show that an antichain is boolean if and only if it spans a lattice that is boolean in L.

Upshots

Theorem 3. *Let L be a finite lattice, d and l integers and $(C_\alpha)_{\alpha\in L}$ be a family of* strong *antichain modules with simple head S_α. If for all $\alpha \in L$ it holds that $\mathbb{S}^l(C_\alpha) \simeq C_\alpha[d]$, then L is $\frac{n}{m}$- fractionally Calabi–Yau.*

Theorem 4. *Let C be a* boolean *antichain of a lattice L. Let $I \subseteq L$ be an interval. There exists at most one integer p such that $\operatorname{Hom}_{D^b}(M_C, I[p])$ is non zero. When such an integer exists, the hom space is one dimensional.*