Crystal Chute Moves on Pipe Dreams

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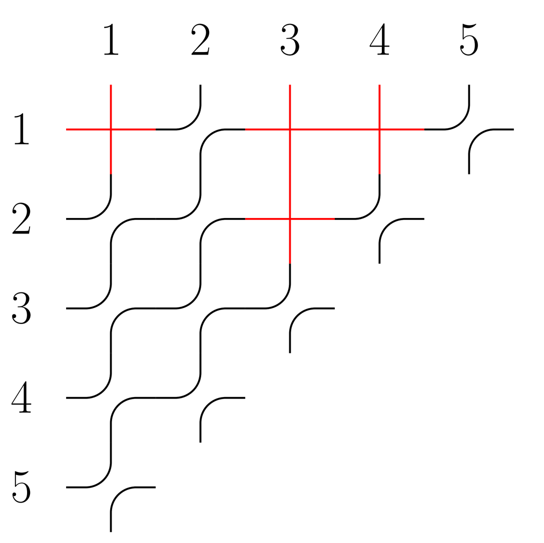
Schubert Polynomials

- $\mathfrak{S}_w$ for $w \in S_n$ is a basis for $H^*(Fl_n)$
- Defined recursively by [Lascoux and Schützenberger 1985]
- Monomials are generated by, for example:
  - rc-graphs [Billey, Jockusch, and Stanley 1993]
  - planar histories [Fomin and Kirillov 1996]
  - reduced pipe dreams [Knutson and Miller 2005]

Pipe Dreams

Combinatorial diagrams generated by permutations in S_n



A pipe dream for $w = [21543]$

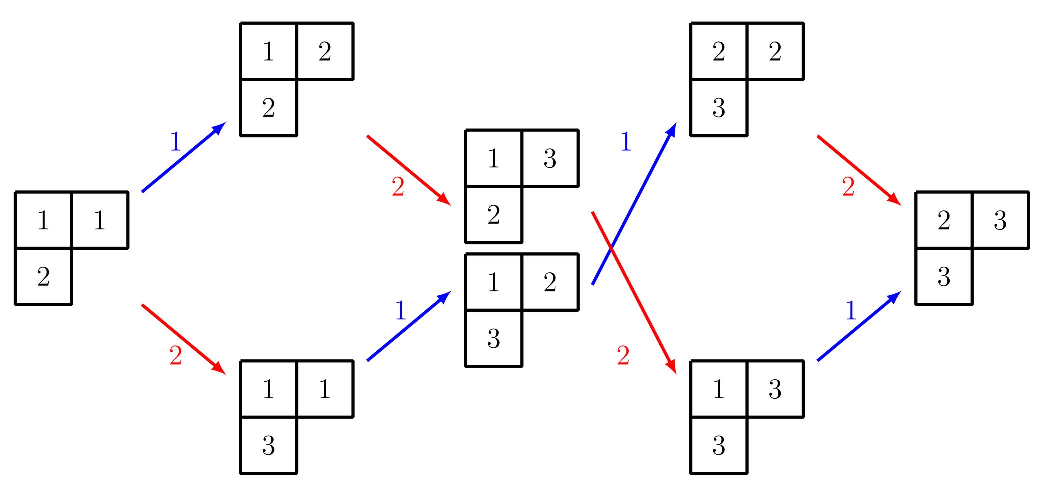
- Each space has a cross or an elbow
- Following flow from the side to the top gives a permutation
- Reduced pipe dreams: each pipe crosses once at most

Theorem 1 (Billey, Jockusch, and Stanley 1993, Fomin and Kirillov 1996, Knutson and Miller 2005)**.** *Let $w \in S_n$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{D \in RP(w)} \mathbf{x}^{\mathrm{wt}(D)}$$

The above pipe dream has weight $(3,1,0,0,0)$, so $\mathfrak{S}_{[21543]}$ has monomial $x_1^3 x_2^1$.

Demazure Crystals



Repeated local moves let us generate all objects from a single highest-weight element

Crystals and Pipe Dreams

Pairing Process on Row i

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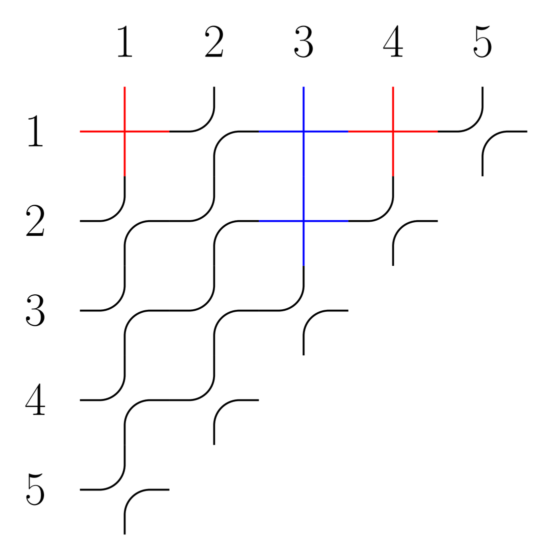
1. Start with right-most unpaired cross in row i
2. Pair with the closest unpaired cross in row $i+1$ weakly southeast of it
3. Move to the next cross to the left in row i and repeat steps 1 and 2.

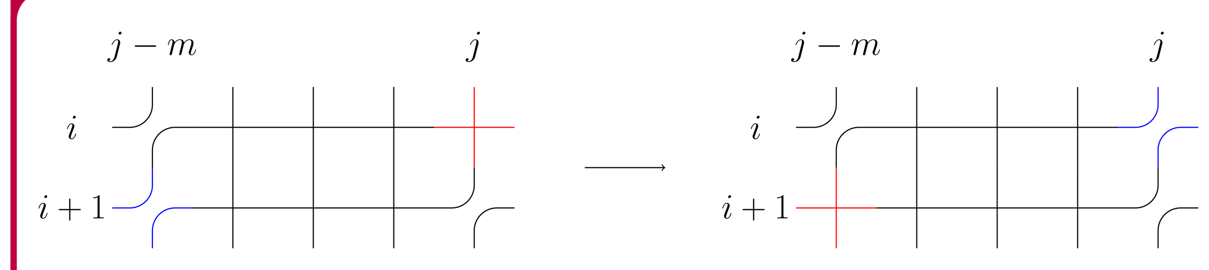


Consider the pairing process on Row 1:

- Cross at $(1,4)$ is unpaired since no crosses are weakly southeast of it
- Cross at $(1,3)$ is paired with cross at $(2,3)$
- Cross at $(1,1)$ is unpaired since no *unpaired* cross is southeast of it

Summary: one paired cross (in blue) and two unpaired crosses (in red) in row 1.

Chute Moves



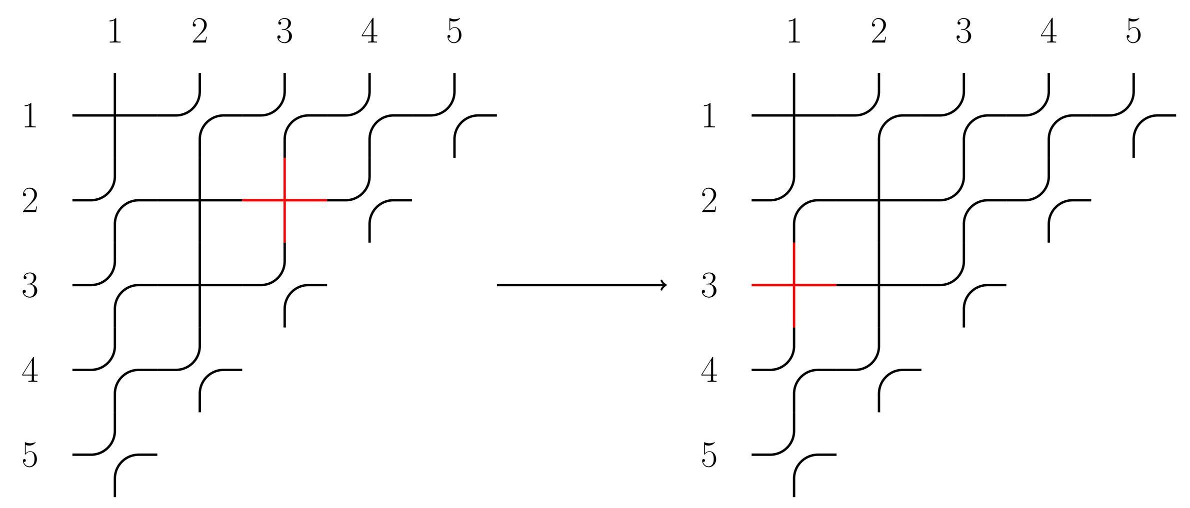
Chute Moves [Bergeron and Billey 1993]: Take a cross at the corner of a rectangle of crosses surrounded on all sides by elbows and move it across the diagonal of that rectangle.

Theorem 2 (Bergeron and Billey 1993). *We can obtain the entirety of $\mathfrak{S}_w$ by applying all chute moves to a single pipe dream D_{top}.*

Lowering Operators on Pipe Dreams

Defining a Lowering Operator f_i

1. Run the pairing process on row i.
2. If all crosses in row i are paired, set $f_i(D) = 0$.
3. Otherwise, take the leftmost unpaired cross in row i and move it via a chute move. The resulting diagram is $f_i(D)$.



Crystal Chute Moves

Theorem 3 (Gold, Milićević, and Sun 2024)**.** *Let $w \in S_n$. The operators e_i, f_i for $1 \le i < n$ define a Type A_{n-1} Demazure crystal structure on the set of reduced pipe dreams* $\mathrm{RP}(w)$*. More precisely,*

$$\mathrm{RP}(w) = \bigcup_{\substack{D \in \mathrm{RP}(w) \\ e_i(D)=0, \forall 1 \le i < n}} B_{\pi_D}(\mathrm{wt}(D))$$

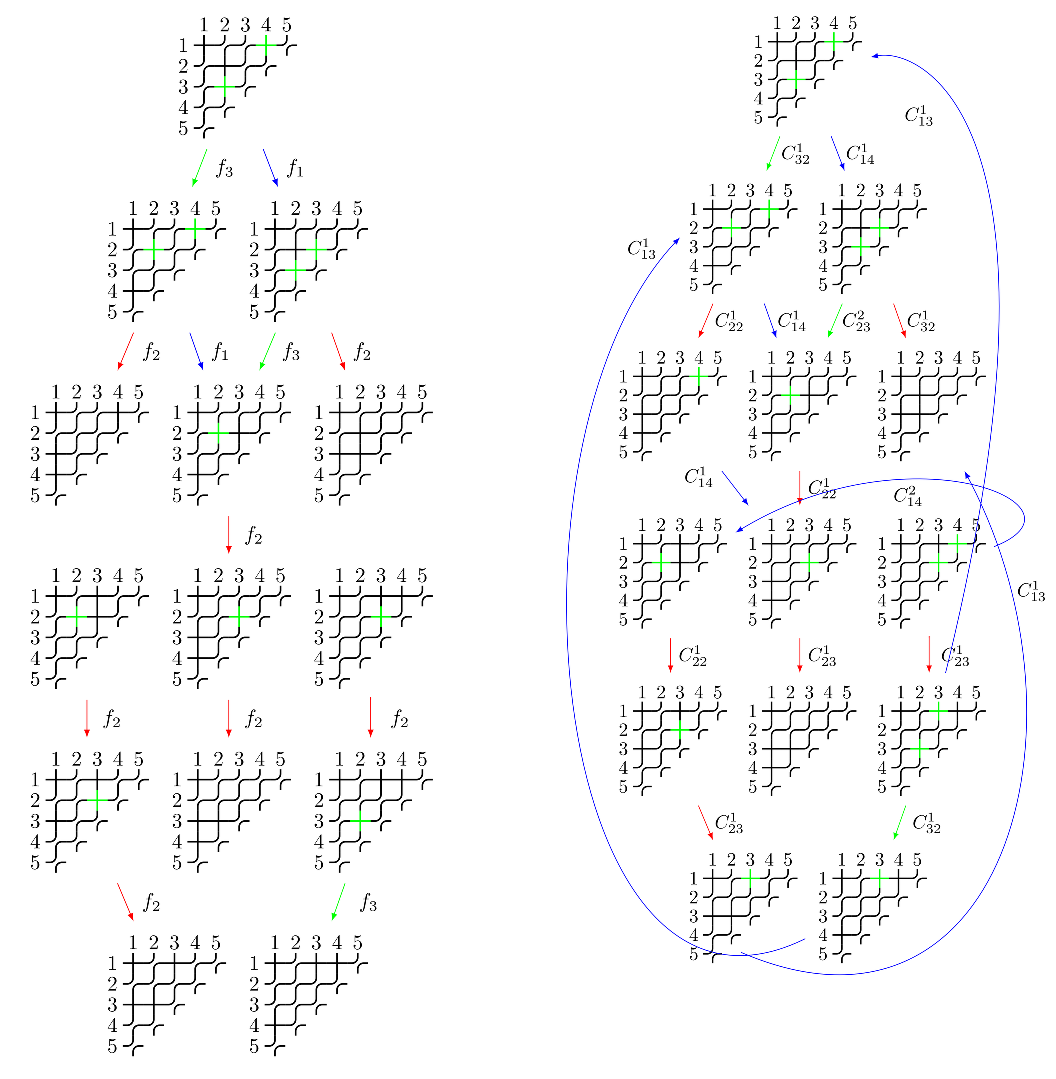
where π_D is a permutation uniquely determined by D.

Corollary 4 (Gold, Milićević, and Sun 2024)**.** *Let $w \in S_n$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\substack{D \in \mathrm{RP}(w) \\ e_i(D)=0, \forall 1 \le i < n}} \kappa_{a_D}(x_1, \ldots, x_n)$$

where a_D is the weak composition $\pi_D(\mathrm{wt}(D))$ and κ_{a_D} is the key polynomial indexed by the weak composition a_D.

Utilizing the Crystal



- Three separate connected components correspond to three key polynomials
- Crystal operators used indicate sorting permutation
- Structure mirrors Demazure crystal structure on key tableaux [Assaf and Schilling 2018]

References

Assaf, Sami and Anne Schilling (2018). "A Demazure crystal construction for Schubert polynomials". In: *Algebraic Combinatorics* 1.2, pp. 225–247.

Bergeron, Nantel and Sara Billey (1993). "RC-graphs and Schubert polynomials". In: *Experiment. Math.* 2.4, pp. 257–269.

Billey, Sara, William Jockusch, and Richard P. Stanley (1993). "Some combinatorial properties of Schubert polynomials". In: *Journal of Algebraic Combinatorics* 2.4, pp. 345–374.

Fomin, Sergey and Anatol N. Kirillov (1996). "The Yang-Baxter equation, symmetric functions, and Schubert polynomials". In: *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993).* Vol. 153. 1-3, pp. 123–143.

Gold, Sarah, Elizabeth Milićević, and Yuxuan Sun (2024). "Crystal Chute Moves on Pipe Dreams". Preprint at `arXiv:2403.07204`.

Knutson, Allen and Ezra Miller (2005). "Gröbner geometry of Schubert polynomials". In: *Ann. of Math. (2)* 161.3, pp. 1245–1318.

Lascoux, Alain and Marcel-Paul Schützenberger (1985). "Schubert polynomials and the Littlewood-Richardson rule". In: *Letters in Mathematical Physics. A Journal for the Rapid Dissemination of Short Contributions in the Field of Mathematical Physics* 10.2-3, pp. 111–124.