Combinatorial properties of triangular partitions

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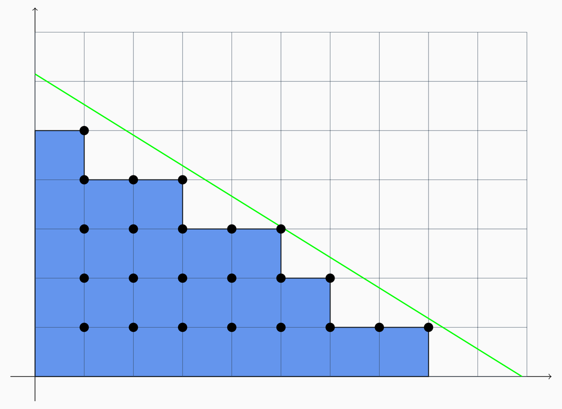
Definition and characterizations

Definition
An integer partition $\tau = \tau_1\tau_2\ldots\tau_k$ is **triangular** if its Ferrers diagram consists of the points in $\mathbb{N}^2$ that lie on or below the line that passes through $(0,s)$ and $(r,0)$ for some $r,s \in \mathbb{R}_{>0}$, called a **cutting line**.



Δ = set of all triangular partitions,
$\Delta(n)$ = set of triangular partitions of size n.

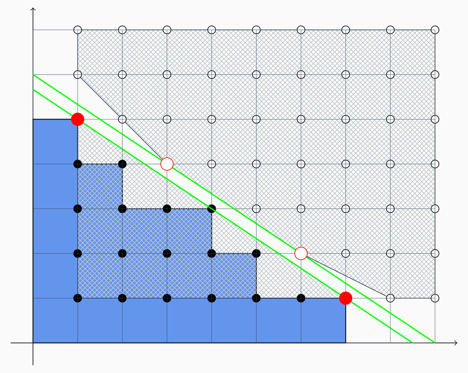
Definition
A cell of $\tau \in \Delta$ is **removable** if removing it from τ yields a triangular partition. A cell of the complement $\mathbb{N}^2 \setminus \tau$ is **addable** if adding it to τ yields a triangular partition.

Lemma (Bergeron, Mazin [2])
Every nonempty triangular partition has either one removable cell and two addable cells, two removable cells and one addable cell, or two removable cells and two addable cells.

$\mathrm{Conv}(S)$ = convex hull of $S \subseteq \mathbb{N}^2$.

Proposition ([4])
A partition λ is triangular if and only if $\mathrm{Conv}(\lambda) \cap \mathrm{Conv}(\mathbb{N}^2 \setminus \lambda) = \emptyset$.

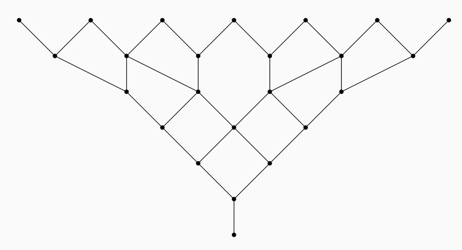
Proposition ([4])
Two cells in $\tau \in \Delta$ are removable if and only if they are consecutive vertices of $\mathrm{Conv}(\tau)$ and the line passing through them does not intersect $\mathrm{Conv}(\mathbb{N}^2 \setminus \tau)$.



The triangular Young poset

$\mathbb{Y}_\Delta$ = poset of triangular partitions ordered by containment of their Ferrers diagrams.

Lemma (Bergeron, Mazin [2])
Let $\tau, \nu \in \mathbb{Y}_\Delta$ such that $\tau < \nu$. Then, $\tau \lessdot \nu$ if and only if τ is obtained from ν by removing exactly one cell. In particular, $\mathbb{Y}_\Delta$ is ranked by the size of the partitions.



Lemma (Bergeron, Mazin [2])
The poset $\mathbb{Y}_\Delta$ has a planar Hasse diagram, and it is a lattice.

Proposition ([4])
The join and the meet of $\tau, \nu \in \mathbb{Y}_\Delta$ are given by

$$\tau \vee \nu = \mathbb{N}^2 \cap \mathrm{Conv}(\tau \cup \nu) \quad \text{and} \quad \tau \wedge \nu = \mathbb{N}^2 \setminus \left(\mathbb{N}^2 \cap \mathrm{Conv}\left(\mathbb{N}^2 \setminus (\tau \cap \nu)\right)\right).$$

Theorem ([4])
Let $\tau, \nu \in \mathbb{Y}_\Delta$ such that $\tau \le \nu$. The value of the Möbius function is:

$$\mu(\tau,\nu) = \begin{cases} 1 & \text{if either } \tau = \nu \text{ or } \exists\, \zeta^1 \neq \zeta^2 : \nu = \zeta^1 \vee \zeta^2,\ \tau \lessdot \zeta^1, \zeta^2, \\ -1 & \text{if } \tau \lessdot \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Bijections to balanced words

Definition
A binary word $w = w_1 \ldots w_\ell$ is **balanced** if for any $h \le \ell$ and $i, j \le \ell - h + 1$,

$$|(w_i + w_{i+1} + \cdots + w_{i+k-1}) - (w_j + w_{j+1} + \cdots + w_{j+k-1})| \le 1.$$

$\mathcal{B}_\ell$ = set of balanced words of length ℓ.

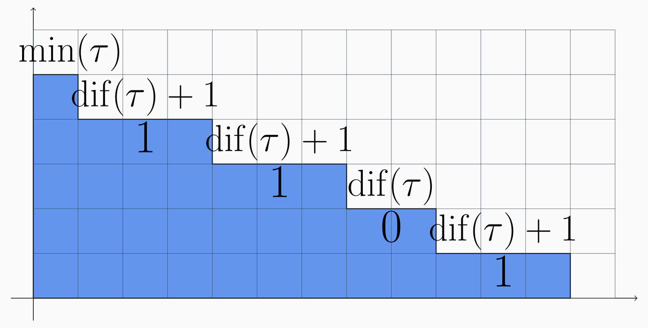
Theorem (Lipatov [5])
The number of balanced words of length ℓ is $|\mathcal{B}_\ell| = 1 + \sum_{i=1}^{\ell}(\ell - i + 1)\varphi(i)$.

A triangular partition is **wide** if all its parts are distinct. Given a wide triangular partition $\tau = \tau_1 \ldots \tau_k$, define the binary word

$$\omega(\tau) = 10^{\tau_1 - \tau_2 - 1}10^{\tau_2 - \tau_3 - 1}\ldots 10^{\tau_{k-1} - \tau_k - 1}10^{\tau_k - 1}.$$

Proposition ([4])
For every $k, \ell \ge 1$, the map ω is a bijection between the set of wide triangular partitions with k parts and first part equal to ℓ, and the set of balanced words of length ℓ with k ones that start with 1.

$\mathcal{B}^0$ = set of balanced words with at least one 0. For $\tau = \tau_1 \ldots \tau_k \in \mathcal{W}$, define
$\mathcal{D}(\tau) = \{\tau_1 - \tau_2,\ \tau_2 - \tau_3,\ \ldots,\ \tau_{k-1} - \tau_k\}$,
$\min(\tau) = \tau_k$, $\mathrm{dif}(\tau) = \min \mathcal{D}(\tau)$,
$\mathrm{wrd}(\tau) = w_1 \ldots w_{k-1}$,
where $w_i = \tau_i - \tau_{i+1} - \mathrm{dif}(\tau)$.



Theorem ([4])
The map $\chi = (\min, \mathrm{dif}, \mathrm{wrd})$ is a bijection between $\mathcal{W}$ and the set

$$\mathcal{T} = \{(m,d,w) \in \mathbb{N} \times \mathbb{N} \times \mathcal{B}^0 \mid m \le d+1;\ w1 \in \mathcal{B}^0 \text{ if } m = d+1\}. \quad (1)$$

Its inverse is given by the map

$$\xi(m,d,w_1\ldots w_{k-1}) = \tau_1 \ldots \tau_k, \quad \text{where } \tau_i = m + \sum_{j=i}^{k-1}(w_j + d) \text{ for } i \in [k].$$

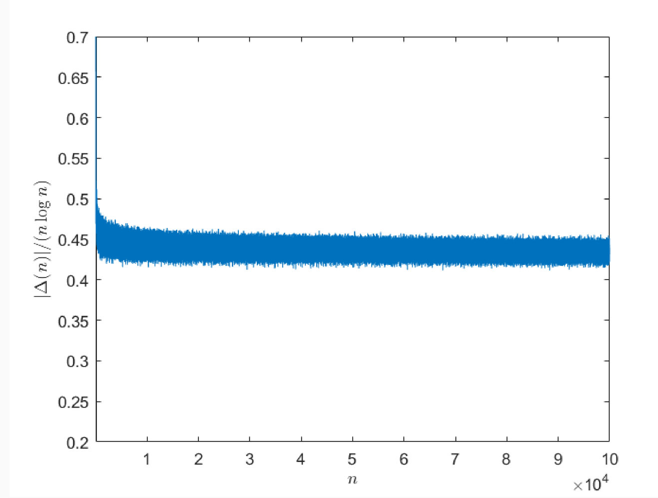
Additionally, given $\tau \in \mathcal{W}$ with image $\chi(\tau) = (m,d,w)$, its size is

$$|\tau| = km + \binom{k}{2}d + \sum_{i=1}^{k-1} i w_i. \quad (2)$$

Efficient enumeration

Enumeration algorithm to compute $|\Delta(n)|$ for $1 \le n \le N$ (available at [1]):

1. On input N, run a depth first search through the tree of balanced words of length up to $\lfloor\sqrt{2N}\rfloor$.
2. For each word, find all values $m, d \in \mathbb{N}$ such that $(m,d,w) \in \mathcal{T}$, as defined in (1), and such that the size function in (2) is at most N.
3. Each triplet (m,d,w) accounts for triangular partition $\tau = \chi(m,d,w)$ and its conjugate.



Complexity: $\mathcal{O}(N^{5/2})$. It allows us to compute the first 10^5 values of $|\Delta(n)|$, compared to the 39 terms known previously. The plot shows $|\Delta(n)|/(n \log n)$, giving experimental evidence for the following result:

Theorem (Corteel et al. [3])
There exist c, c' such that $cn\log n < |\Delta(n)| < c'n \log n$ for all $n > 1$.

Triangular subpartitions

$I(\sigma^\ell)$ = number of triangular subpartitions of the staircase of ℓ parts.
$\Delta^{\ell \times \ell}$ = set of triangular partitions inside a square of side ℓ.

Lemma ([4])
The number of triangular partitions of width exactly ℓ and height at most ℓ is $|\mathcal{B}_\ell|/2$, and

$$\left|\Delta^{\ell\times\ell} \setminus \Delta^{(\ell-1)\times(\ell-1)}\right| = I(\sigma^\ell) - I(\sigma^{\ell-1}) = |\mathcal{B}_\ell| - 1.$$

Theorem ([4])

$$\left|\Delta^{\ell\times\ell}\right| = I(\sigma^\ell) = 1 + \sum_{i=1}^{\ell}\binom{\ell - i + 2}{2}\varphi(i).$$

We also have a direct combinatorial proof of this theorem, from which we derive as a byproduct a new proof of Lipatov's enumeration theorem for balanced words.

References

[1] Code available at https://math.dartmouth.edu/~sergi/tp.
[2] François Bergeron and Mikhail Mazin. "Combinatorics of triangular partitions". In: *Enumer. Comb. Appl.* 3.1 (2023), Paper No. S2R1, 20.
[3] Sylvie Corteel, Gaël Rémond, Gilles Schaeffer, and Hugh Thomas. "The Number of Plane Corner Cuts". In: *Advances in Applied Mathematics* 23.1 (1999), pp. 49–53.
[4] Sergi Elizalde and Alejandro B Galván. "Triangular partitions: enumeration, structure, and generation". In: *arXiv preprint arXiv:2312.16353* (2023).
[5] E. P. Lipatov. "On a classification of binary collections and properties of homogeneity classes". Russian. In: *Probl. Kibern.* 39 (1982), pp. 67–84.

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