Excedance quotients, Quasisymmetric Varieties, and Temperley—Lieb algebras

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Quasisymmetric Polynomials & Hivert's Action

Let $R_n = \mathbb{C}[x_1, x_2, \dots, x_n]$. For any subset $I = \{i_1 < i_2 < \cdots < i_\ell\} \subseteq [n]$ and composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ of the same length, define

$$X_I^\alpha = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \quad \text{e.g.} \quad x_{\{1,2,4\}}^{(2,1,1)} = x_1^2 x_2 x_4.$$

Hivert has defined a (non-standard) action "$*$" of S_n on R_n:

$$w * X_I^\alpha = X_{w(I)}^\alpha \quad \text{e.g.} \quad 2143 * x_1^2 x_2 x_4 = x_1^2 x_2 x_3$$

The *quasisymmetric polynomials* are the fixed points of $*$: $\mathsf{QSym}_n = R_n^{*S_n}$. For example,

$$M_{(2,1,1)} = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + x_2^2 x_3 x_4 \in \mathsf{QSym}_4$$

Key

● ● $\in S_n$

● $\in \mathsf{QSV}_n$

Bruhat Order:

The Bruhat order is the partial order on S_n generated by the relation:

$u \lessdot v$ if and only if $(a\,b)u = v$ and $\ell(u) = \ell(v) - 1$

The Excedance Relation

A *weak excedance* of $w = w_1 w_2 \cdots w_n \in S_n$ is a pair (i, w_i) with $i \le w_i$. Let

$$E_{pos}(w) = \{i \mid i \le w_i\} \quad \text{and} \quad E_{val}(w) = \{w_i \mid i \le w_i\}.$$

We define a novel equivalence relation on S_n:

$$w \sim v \text{ if and only if } E_{pos}(w) = E_{pos}(v) \text{ and } E_{val}(w) = E_{val}(v)$$

$$w = \begin{smallmatrix} \underline{1} & \underline{2} & \underline{3} & 4 & \underline{5} & 6 & 7 \\ \underline{5} & \underline{2} & \underline{6} & 1 & \underline{7} & 4 & 3 \end{smallmatrix} \qquad E_{pos}(w) = \{\underline{1}, \underline{2}, \underline{3}, \underline{5}\} = E_{pos}(v) \qquad \begin{smallmatrix} \underline{1} & \underline{2} & \underline{3} & 4 & \underline{5} & 6 & 7 \\ \underline{2} & \underline{7} & \underline{6} & 3 & \underline{5} & 1 & 4 \end{smallmatrix} = v$$

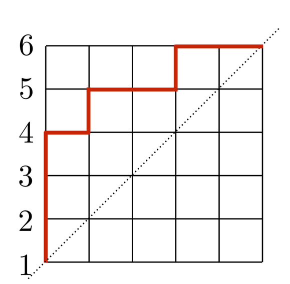
$$E_{val}(w) = \{\underline{2}, \underline{5}, \underline{6}, \underline{7}\} = E_{val}(v)$$

Two Catalan Quotients

Constructing QSym_n gives *two* quotients with dimension

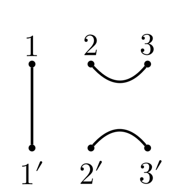
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

(i) The *coinvariant space* $R_n / \langle \mathsf{QSym}_n^+ \rangle$: Aval–Bergeron–Bergeron compute the initial ideal find a basis of C_n (standard) monomials.

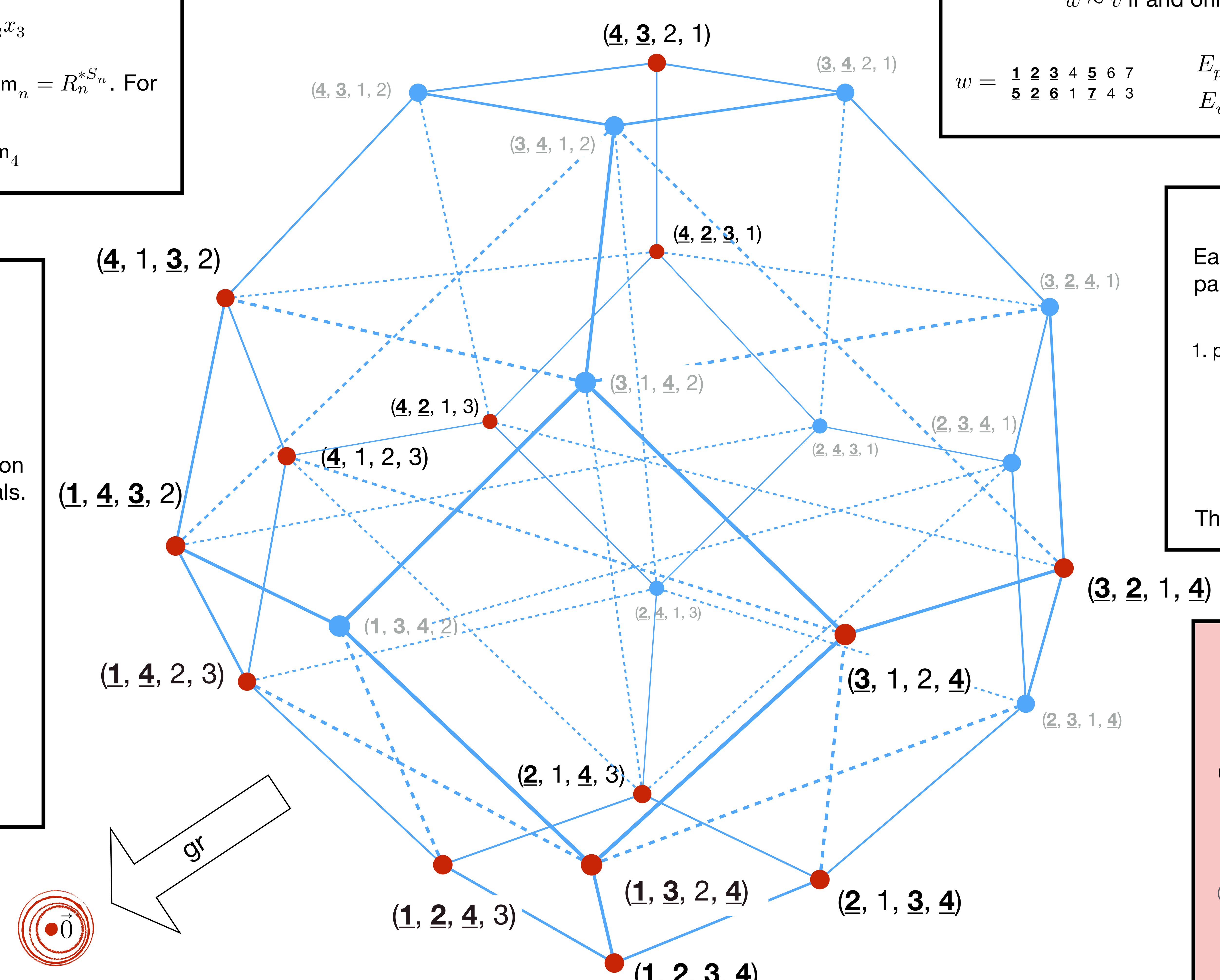


$\longleftrightarrow \quad x_4 x_5^2 + \langle \mathsf{QSym}_n^+ \rangle$

(ii) The *Temperley–Lieb algebra* $\mathsf{TL}_n(2) \cong \mathbb{C}S_n / \ker(*)$ has a famous basis of (C_n) non-crossing matchings on $2n$ vertices.



$= 132 - 123 + \ker(*) \in \mathsf{TL}_3(2)$



gr

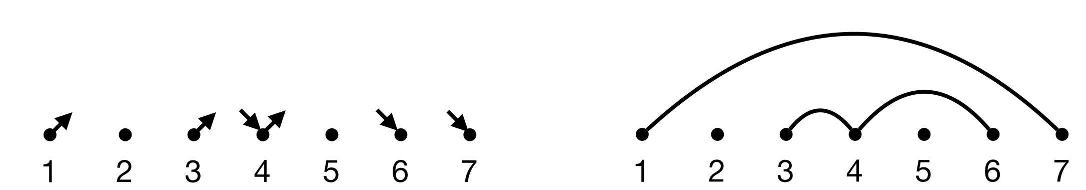
$R_n / \langle \mathsf{QSym}_n^+ \rangle$

Noncrossing Partitions

Each class in $S_n/\sim$ determines a unique noncrossing partition of $[n]$:

1. place: ↗ at each $i \in [n] - E_{val}$; ↘ at each $i \in [n] - E_{pos}$

2. (uniquely) connect ↗ to ↘ without creating crossings

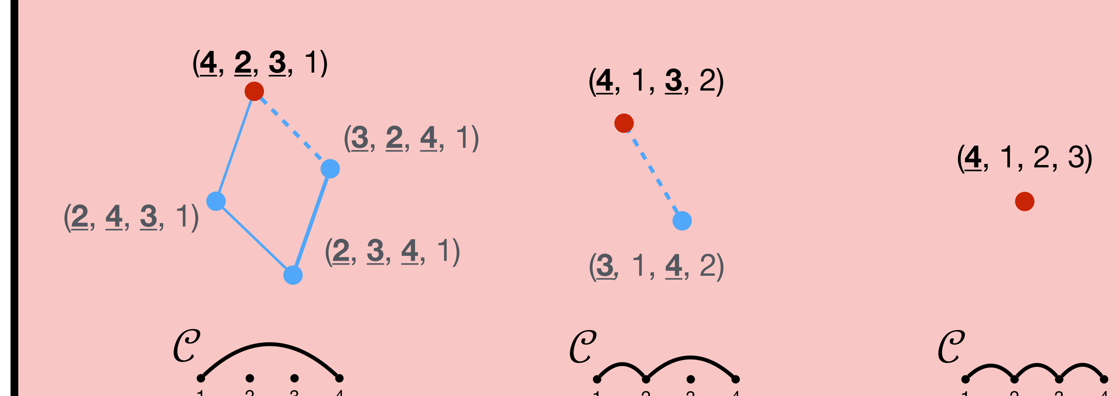


The number of noncrossing partitions is C_n.

Excedance Classes

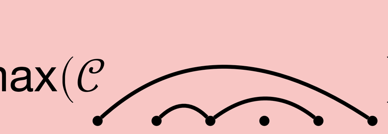
Theorem [B.–G.]: For each noncrossing partition λ of $[n]$, let $\mathcal{C}_\lambda \in S_n/\sim$ be the class corresponding to λ.

(i) $\mathcal{C}_\lambda$ is a non-empty interval in the Bruhat order,



(ii) the Bruhat-min of $\mathcal{C}_\lambda$ is 321-avoiding,

(iii) the Bruhat-max of $\mathcal{C}_\lambda$ is a "noncrossing partition,"

max() $= (6\,1)\,(5\,3\,2)\,(4) = 652431$

(iv) restricting the Bruhat order to $S_n/\sim$ gives an order isomorphic to a dual interval of Young's lattice (similar to a result of Gobet–Williams), and

(v) any section $(w_\lambda \in \mathcal{C}_\lambda)_{\text{noncrossing } \lambda}$ descends modulo $\ker(*)$ to a basis of $\mathsf{TL}_n(2)$.

Comparison with Sym

For the standard S_n-action "$\cdot$" on polynomials, $R_n^{\cdot S_n} = \mathsf{Sym}_n$, the symmetric polynomials.

(i) The dimension of $R_n / \langle \mathsf{Sym}_n^+ \rangle$ is $n!$.

(ii) $\ker(\cdot) = 0$, so $\dim(\mathbb{C}S_n / \ker(\cdot)) = n!$.

(iii) $R_n / \langle \mathsf{Sym}_n^+ \rangle$ canonically affords the regular representation of S_n.

Aval–Bergeron–Bergeron speculated that a version of (iii) holds for QSym_n, but there is no canonical action as $\langle \mathsf{QSym}_n^+ \rangle$ is not fixed by $*$.

gr ("Orbital Harmonics")

For $f \in R_n$, let $\mathsf{h}(f) =$ top degree part of f.

$$\mathsf{h}(x_1^2 x_3 + x_1 - 7) = x_1^2 x_3.$$

For $I \subseteq R_n$, let $\mathsf{gr}(I) = \langle \mathsf{h}(f) \mid f \in I \rangle$. Then $\mathsf{gr}(I)$ is homogeneous and

$$R_n / I \cong R_n / \mathsf{gr}(I)$$

as *vector spaces* (think: I is the vanishing ideal of a variety of points).

The Quasisymmetric Variety $\mathsf{QSV}_n \subseteq S_n$

Theorem [B.–G.]: Let $\mathsf{QSV}_n = \{\max(\mathcal{C}_\lambda) \mid \text{noncrossing partitions } \lambda \text{ of } [n]\}$.

(i) $\mathsf{gr}(\mathbf{I}(\mathsf{QSV}_n)) = \langle \mathsf{QSym}_n^+ \rangle$, where $\mathbf{I}(-)$ denotes the vanishing ideal,

(ii) as vector spaces, $R_n / \langle \mathsf{QSym}_n^+ \rangle \cong R_n / \mathbf{I}(\mathsf{QSV}_n)$, and

(iii) QSV_n descends to a basis of $\mathbb{C}S_n / \ker(*)$, so $\mathsf{TL}_n(2) \cong R_n / \mathbf{I}(\mathsf{QSV}_n)$.

This gives a natural action of $\mathsf{TL}_n(2)$ on $R_n / \langle \mathsf{QSym}_n^+ \rangle$.

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