# Shi arrangements and low elements in Coxeter groups

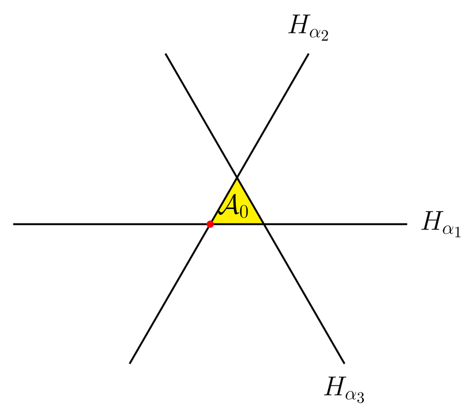
Matthew Dyer[1] & Susanna Fishel[2†] & Christophe Hohlweg[3⋆] & Alice Mark[4]

[1]University of Notre Dame, [2]Arizona State University, [3]Université du Québec à Montréal, [4]Vanderbilt University

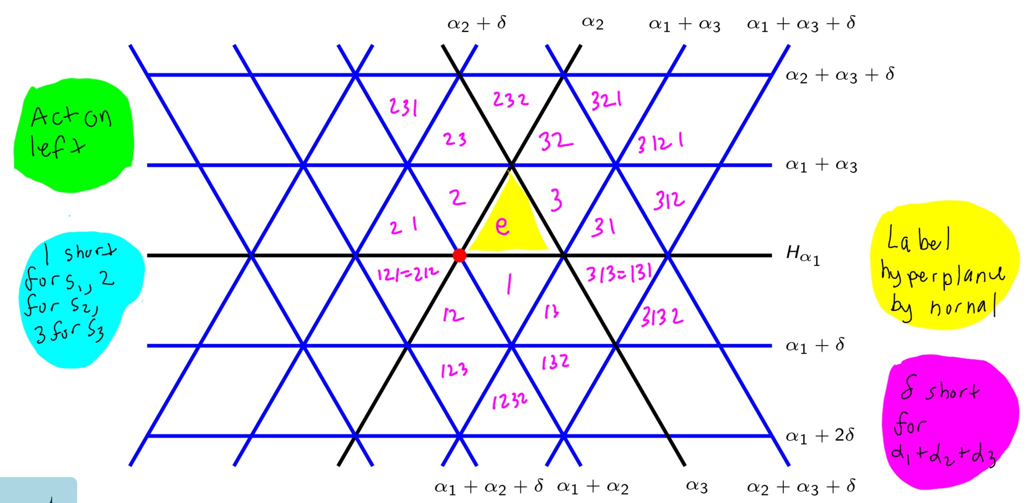
## Classical Shi arrangement

### Affine $\mathcal{S}_3$

$\tilde{A}_2$ generated by $s_1, S_2, s_3$, which are reflections over the hyperplanes $H_{\alpha_1}, H_{\alpha_2}, H_{\alpha_3}$.

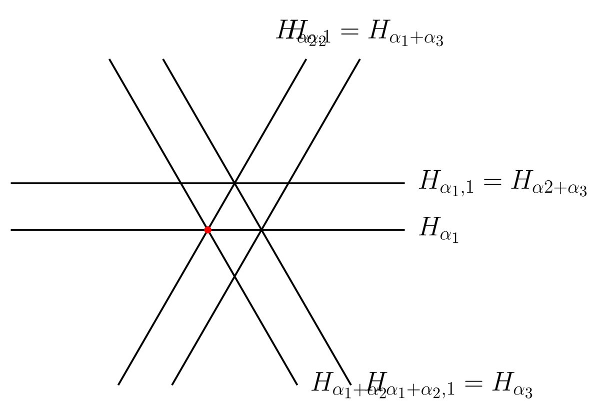



### Alcoves and elements of affine $\mathcal{S}_3$



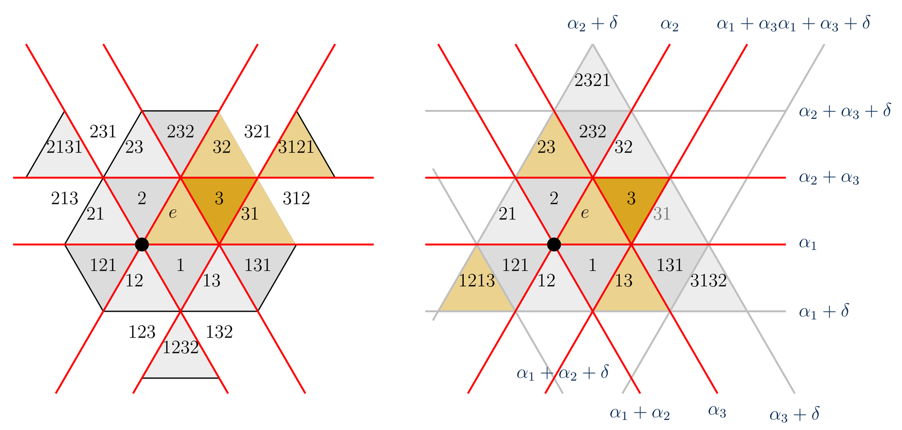

### Shi arrangement

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$



### Wonderful properties

Each Shi region has a unique element closest to the origin and the collection of their inverses forms a simplex. We identify an element of the group with its alcove.



**What about non-affine, non-finite Coxeter groups?**

## Other Coxeter groups

### Coxeter system

A Coxeter system is a pair $(W, S)$

- $W$ is a group
- $S \subset W$ is a set of generators of $W$ ($S$ will be finite in this poster)

subject to the following: for every pair of generators $s$ and $s'$, we have an integer $m(s, s')$ such that $(ss')^{m(s,s')} = e$, $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in $S$.

In our example in column 1, $m(s_i, s_j) = \begin{cases} 3 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\quad s_is_i = e$ since $s_i$ is a reflection

### Standard tools for the combinatorics of Coxeter Groups

$w \in W$, $s_{i_j} \in S$, $w = s_{i_1}s_{i_2}\cdots s_{i_k}$

- LENGTH: If $k$ is minimal, then $s_{i_1}s_{i_2}\cdots s_{i_k}$ is a reduced expression for $w$ and the length of $w$ is $k$. Write $\ell(w) = k$. The length is the minimum number of hyperplanes crossed when travelling from $\mathcal{A}_0$ to $w\mathcal{A}_0$
- INVERSIONS: $\Phi(w) = \Phi^+ \cap w\Phi^- = \{\beta \in \Phi^+ | \ell(s_\beta w) < \ell(w)\}$ The inversions are the labels of the hyperplanes crossed when travelling from $\mathcal{A}_0$ to $w\mathcal{A}_0$
- RIGHT DESCENTS: $\mathrm{Des}_R(w) = \{s \in S | \ell(ws) < \ell(w)\}$ $ws\mathcal{A}_0$ is the neighbor of $w\mathcal{A}_0$
- RIGHT DESCENT ROOTS: $\Phi^R(w) = \{-w(\alpha_s) | s \in \mathrm{Des}_R(w)\}$ In bijection with $\mathrm{Des}_R(w)$, they are the normals to the boundary hyperplanes for alcove $w\mathcal{A}_0$
- LEFT DESCENT: $\mathrm{Des}_L(w) = \{s \in S | \ell(sw) < \ell(w)\}$
- LEFT DESCENT ROOTS: $\Phi^L(w) = \Phi(w) \cap \Delta$

### Dominance order on roots

The dominance order [1] is the partial order $\preceq_{\mathrm{dom}}$ on $\Phi^+$ defined as follows:

$$\alpha \preceq_{\mathrm{dom}} \beta \iff (\forall w \in W,\ \beta \in \Phi(w) \implies \alpha \in \Phi(w)).$$

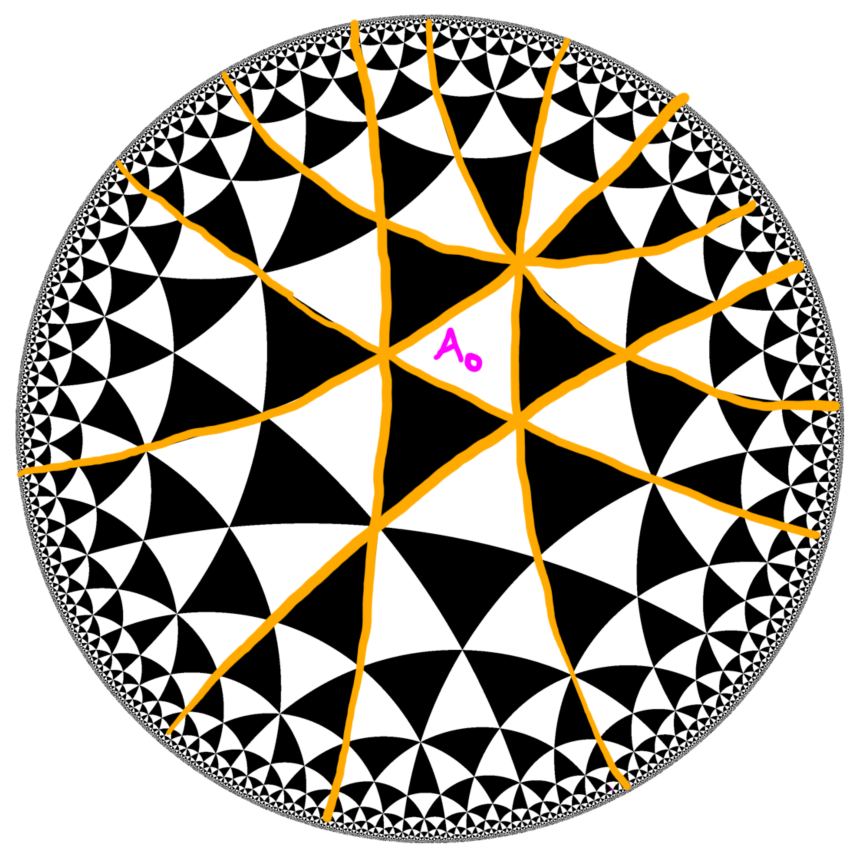
$\alpha \preceq_{\mathrm{dom}} \beta$ if for every $w$: if I have to hop over $H_\beta$ to get to $w\mathcal{A}_0$, then I had to hop over $H_\alpha$. Also, dominance can be extended to all roots.

### Small roots

The set of small roots is the set of positive roots that do not dominate any positive roots.

### Example

$\tilde{A}_2 : \Sigma_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$, normals to the Shi hyperplanes



Seven small roots for the triangle group (3, 3, 4)

### Shi arrangement and low elements

The Shi arrangement [4] $\mathrm{Shi}_0(W, S)$ is the arrangement of small hyperplanes:

$$\mathrm{Shi}_0(W, S) = \{H_\beta \mid \beta \in \Sigma_0\},$$

The small inversion set of $w \in W$ is the set:

$$\Sigma_0(w) = \Phi(w) \cap \Sigma_0.$$

The set $L_0$ of low elements is

$$L_0 = \{w \in W \mid \Phi^1(w) \subseteq \Sigma_0\}.$$

### Conjecture and theorem

Conjecture from Dyer and Hohlweg[3]

**Theorem 1: Mininmal elements are low**

Let $(W, S)$ be a Coxeter system.

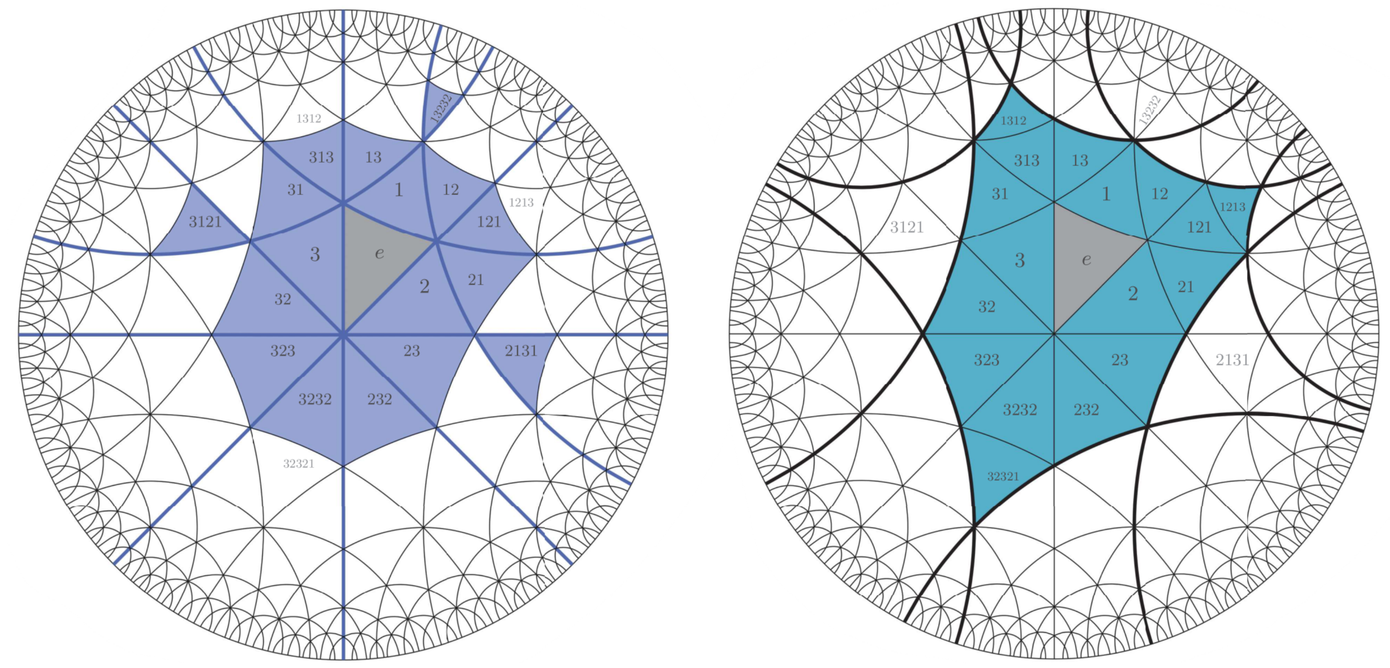
1. Each region of $\mathrm{Shi}_0(W, S)$ contains a unique element of minimal length.
2. The set of the minimal length elements of $\mathrm{Shi}_0(W, S)$ is equal to the set $L_0$ of low elements.

Conjecture from Hohlweg, Nadeau, Williams[4].

**Theorem 2: Low element inverses form a convex set**

Let $(W, S)$ be a Coxeter system. The union of the alcoves $w^{-1}\mathcal{A}_0$ for $w \in L_0$ is a convex set.

### Example



## Highlights from the proof [2]

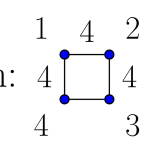
### Short inversion posets

Let $w \in W$ with short inversions $\alpha, \beta \in \Phi^1(w)$. We write $\alpha \prec_w \beta$ if $\beta$ is not in the simple system for the maximal dihedral subgroupl containing the subgroup generated by $s_\alpha$ and $s_\beta$, but $\alpha$ is. Take the transitive and reflexive closure to form a partial order.

**Theorem 3: Sandwich theorem**

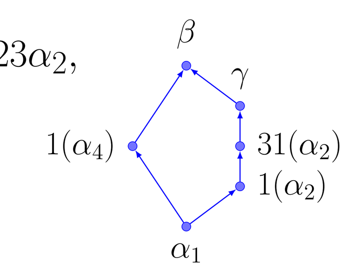
Let $w \in W$. For the poset $(\Phi^1(w), \preceq_w)$, the minimal elements are the left-descent roots in $\Phi^L(w)$ and the maximal elements are the right-descent roots in $\Phi^R(w)$. More precisely, for any $\beta \in \Phi^1(w)$ there is $\alpha \in \Phi^L(w)$ and $\gamma \in \Phi^R(w)$ such that $\alpha \preceq_w \beta \preceq_w \gamma$.

Consider $(W, S)$ with $S = \{1, 2, 3, 4\}$ with Coxeter graph:




This is an indefinite Coxeter system, i.e., neither finite nor affine.
Let $w = 1234232314$, then

- $\Phi(w) = \{\alpha_1, 1\alpha_2, 12\alpha_3, 123\alpha_4, 1234\alpha_2, 12342\alpha_3, 123423\alpha_2, 1234232\alpha_3, 12342323\alpha_1, 123423231\alpha_4\}$
- $\Phi^L(w) = \{\alpha_1\}$
- $\Phi^R(w) = \{123432321(\alpha_4)\}$
- $\Phi^1(w) = \{\alpha_1, 1(\alpha_2), 31(\alpha_2), \gamma =, 1(\alpha_4), \beta\}$




### Polytope

Consider the set

$$\mathcal{B}_0 = \{x^{-1}(\alpha_s) \mid x \in L_0,\ s \in S,\ sx \notin L_0\}.$$

The Shi polyhedron is the convex set

$$\mathcal{S}_0 = \bigcap_{\beta \in \mathcal{B}_0} H^+_\beta.$$

[1] Brigitte Brink and Robert B. Howlett. A finiteness property and an automatic structure for Coxeter groups. *Math. Ann.*, 296:179–190, 1993.

[2] Matthew Dyer, Christophe Hohlweg, Susanna Fishel, and Alice Mark. Shi arrangements and low elements in coxeter groups, 2024. To appear in the Proceedings of the London Mathematics Society.

[3] Matthew J. Dyer and Christophe Hohlweg. Small roots, low elements, and the weak order in Coxeter groups. *Advances in Mathematics*, 301:739–784, 2016.

[4] Christophe Hohlweg, Philippe Nadeau, and Nathan Williams. Automata, reduced words and Garside shadows in Coxeter groups. *J. Algebra*, 457:431–456, 2016.

† SF is supported by the Simons Collaboration Grants for Mathematicians 359602,709671. ⋆ CH is supported by the NSERC grant *Algebraic and Geometric Combinatorics of Coxeter groups*.