Enumerating the faces of split matroid polytopes

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Abstract

Computing f-vectors of polytopes is in general a challenging task, and only little is known about their shape. This motivated us to initiate the study of properties of f-vectors of matroid base polytopes, by focusing on the class of split matroids, i.e., matroid polytopes arising from compatible splits of a hypersimplex. Unlike valuative invariants, the f-vector behaves in a much more unpredictable way, and the modular pairs of cyclic flats play a role in the face enumeration. Here we give a concise description of how the computation can be achieved without performing any expensive convex hull or face lattice computation. As applications, we deduce formulas for sparse paving matroids and rank-2 matroids. These are two families that appear in other contexts within combinatorics.

A quick recapitulation

For a convex polytope $\mathscr{P}$, the f-vector is defined as $f(\mathscr{P}) = (f_0, \dots, f_d)$, where $d = \dim \mathscr{P}$ and

$$f_i = \#\{i\text{-dimensional faces of } \mathscr{P}\}.$$

These numbers are often arranged into a polynomial, called the f-polynomial of $\mathscr{P}$,

$$f_{\mathscr{P}}(t) = f_0 + f_1 x + \cdots + f_d\, x^d.$$

Very little is known about f-vectors of polytopes in general, and even so for specific subclasses of polytopes. We are mainly interested in computing this polynomial for a *matroid base polytope* $\mathscr{P}$. We initiate this study by focusing on a large family of matroids which arise from the study of rays of the tropical Grassmannian which are called *split matroids* (see further below for a precise definition).

A matroid M on the ground set $E = [n]$ can be encoded through its base polytope

$$\mathscr{P}(\mathsf{M}) = \text{convex hull}\,\{e_B : \ B \text{ is a basis of } \mathsf{M}\}\,,$$

here $e_B = \sum_{i\in B} e_i$ denotes the indicator vector of the basis B.
If the rank of M is k, it is immediate to check that $\mathscr{P}(\mathsf{M})$ is a subpolytope of the hypersimplex

$$\Delta_{k,n} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \right\}$$

An exterior description of a matroid base polytope is given through its cyclic flats as follows

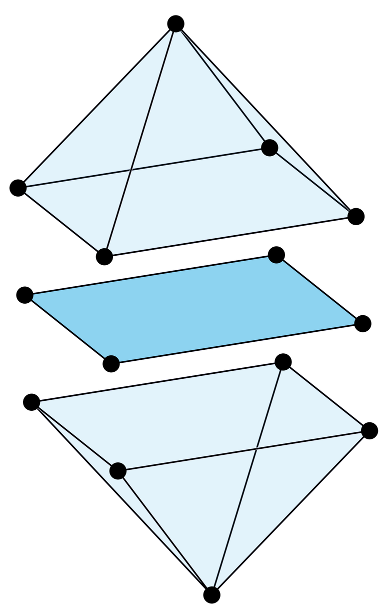
$$\mathscr{P}(\mathsf{M}) = \left\{ x \in \Delta_{k,n} : \sum_{i\in F} x_i \leq \operatorname{rk}(F) \text{ for all cyclic flats } F \right\}.$$

Subdivisions and split matroids

The polytope of a split matroid arises as a cell in a particular matroid subdivision of the hypersimplex. Formally a *matroid subdivision* of a matroid polytope $\mathscr{P}(\mathsf{M})$ is a polyhedral complex all of whose cells are base polytopes of matroids. A subdivision in which there are only two maximal cells which intersect along a common facet is called a *hyperplane split*.

Definition 1 (Joswig-Schröter)**.** A matroid M on n elements and rank k is a *split matroid* if its base polytope $\mathscr{P}(\mathsf{M})$ arises from the hypersimplex $\Delta_{k,n}$ via a sequence of hyperplane splits in which no two of the hyperplanes intersect in the relative interior of $\Delta_{k,n}$.

Below we depict an example of a matroid subdivision and the split matroids arising from it.



Here the full octahedron corresponds to the base polytope of the uniform matroid $\mathsf{U}_{2,4}$. The pyramids and square are base polytopes of a split matroid, because they arise from either a single or two hyperplane splits of the second hypersimplex $\Delta_{2,4}$.

Remark 2. The f-vector does not interact well with matroid subdivisions. Notice that in the figure on the left, we have that the f-polynomial of both pyramids is $5 + 8x + 5x^2 + x^3$, and the f-polynomial of the square in the middle is $4+4x$. If the f-vector would be a 'valuative invariant', one would expect that the f-vector of the full octahedron is $2\cdot(5+8x+5x^2+x^3) - (4+4x) = 6+12x+10x^2+2x^3$, which is clearly not the case (the leading coefficient should always be 1).

The main result

Our main result is the following explicit formula for the f-polynomial of an arbitrary split matroid.

Theorem 6. *Let* M *be a (connected) split matroid of rank k on n elements. The number of faces of its base polytope $\mathscr{P}(\mathsf{M})$ is given by the polynomial*

$$f_{\mathscr{P}(\mathsf{M})}(t) = f_{\Delta_{k,n}}(t) - \sum_{r,h} \lambda_{r,h} \cdot u_{r,k,h,n}(t) - \sum_{\alpha,\beta,a,b} \mu_{\alpha,\beta,a,b} \cdot w_{\alpha,\beta,a,b}(t)$$

where the first sum ranges over all values with $0 < r < h < n$ and the second sum ranges over the values $0 < \alpha < a$, $0 < \beta < b$ for which either $a < b$ or $a = b$ and $\alpha \leq \beta$.

The numbers appearing on the right-hand-side of this formula are, respectively,

- $\lambda_{r,h}$ the number of proper non-empty cyclic flats of size h and rank r in M.
- The expressions u and w are polynomials that only depend on their subindices.
- $\mu_{\alpha,\beta,a,b}$ the number of modular pairs of cyclic flats $\{F_1, F_2\}$ such that $a = |F_1 \smallsetminus F_2|$, $b = |F_2 \smallsetminus F_1|$, $\alpha = \operatorname{rk}(F_1) - \operatorname{rk}(F_1\cap F_2)$ and $\beta = \operatorname{rk}(F_2) - \operatorname{rk}(F_1 \cap F_2)$.

Recall that a modular pair of flats F_1, F_2 in a matroid is a pair of flats satisfying:

$$\operatorname{rk}(F_1) + \operatorname{rk}(F_2) = \operatorname{rk}(F_1\cup F_2) + \operatorname{rk}(F_1 \cap F_2).$$

The above theorem says that the f-vector of a split matroid polytope not only depends on the sizes and ranks of the cyclic flats, but explains precisely what additional piece of information is required to compute it: the intersection data provided by the modular pairs of cyclic flats.

Schubert elementary split matroids

A matroid is said to be a *Schubert matroid* if its lattice of cyclic flats is a chain. For (connected) split matroids we have the following characterization in terms of cyclic flats.

Proposition 3. *A connected matroid* M *is split if and only if its family of proper non-empty cyclic flats forms a clutter (i.e., no two of them are comparable).*

Combining these two observations, we are led to consider the class of Schubert elementary split matroids.

Definition 4. Let $E = [n]$, $\varnothing \neq F \subsetneq E$ and $r \leq k \leq n$. The matroid $\Lambda_{k,n}^F$ is defined as the unique matroid on $[n]$ having rank k and the proper non-empty cyclic flat F of rank r.

The matroids $\Lambda_{k,n}^F$ are exactly the class of matroids that are Schubert and elementary split. In some sources they are called "cuspidal matroids". For concreteness, when $F = \{1, \dots, h\}$, its base polytope can be succinctly described as

$$\mathscr{P}(\Lambda_{k,n}^F) = \left\{ x \in \Delta_{k,n} : \sum_{i=1}^{h} x_i \geq k - r \right\}.$$

This shows indeed that the matroid $\Lambda_{k,n}^F$ is split as it arises via a single hyperplane split of $\Delta_{k,n}$.

Remark 5. The f-vectors of $\Delta_{k,n}$ and $\Lambda_{k,n}^F$ can be computed explicitly via sums of binomial numbers. Concretely,

$$f_{\Delta_{k,n}}(t) = \binom{n}{k} + \sum_{i=1}^{n-1} \binom{n}{i+1} \sum_{j=1}^{i} \binom{n-i-1}{k-j} \cdot t^i \ .$$

A more complicated (but still very concrete) formula can be written down for $\Lambda_{k,n}^F$, but we omit it here and point the interested reader to the extendend abstract.

Our building blocks are in fact the two families of polynomials $u_{r,k,h,n}(t) = f_{\Delta_{k,n}}(t) - f_{\mathscr{P}(\Lambda_{k-r,k,n-h,n})}(t)$ and $w_{\alpha,\beta,a,b}(t) = f_{\mathscr{P}(\Lambda_{\beta,\alpha+\beta,b,a+b})}(t) + f_{\mathscr{P}(\Lambda_{\alpha,\alpha+\beta,a,a+b})}(t) - f_{\Delta_{\alpha+\beta,a+b}}(t) - f_{\Delta_{\alpha,a}}(t)\cdot f_{\Delta_{\beta,b}}(t)$, depending only on their subindices. We provide explicit expressions for these polynomials in terms of binomial sums, but again omit the formulas on this poster.

Some consequences

We can compute f-vectors for a very large class of matroid polytopes without constructing any face lattice nor taking costly convex hulls. Moreover, since split matroids comprise the classes of sparse paving, paving, and rank-2 matroids, we can specialize our formulas to those cases and generalize existing results in the literature.
Furthermore, we state a few open questions that are motivated by the large amount of experiments that we were able to conduct using the main result. One of them is the following.

Question 7. Is it true that there exists a number c such that the first c entries of the f-vector of some matroid M, i.e., $f_0, \dots, f_{c-1}$, are enough to determine the complete f-vector of M? If it is true, what is the smallest such c?

This question is motivated by properties of matroid subdivisions and the reconstructibility proved by Pineda-Villavicencio and Schröter. More questions related to the log-concavity of these f-vectors, and the relation with extension complexity of matroid polytopes is also discussed in the extended abstract and paper.

References:

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