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Geometry of C-Matrices for Mutation-Infinite Quivers

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Forks

A **fork** is an abundant—two or more arrows between every pair of vertices—quiver F, where F is not acyclic and there exists a vertex r, called the point of return, such that

- For all i such that $i \to r$ and j such that $r \to j$ we have $f_{ji} > f_{ir}$ and $f_{ji} > f_{rj}$.
- The full subquiver formed by removing the vertex r from F is acyclic.

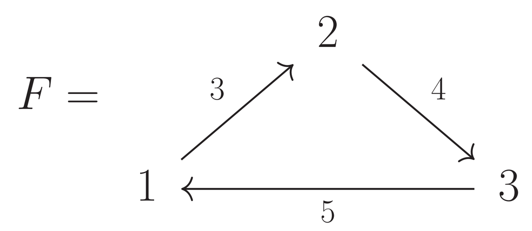

Figure 1. A Fork with Point of Return 2

Every mutation-infinite quiver is mutation-equivalent to a fork, and mutating a fork at any vertex other than the point of return produces another fork [War14]. As such, a **fork-preserving mutation sequence** is any mutation sequence that starts with a fork and does not mutate at any point of return.

C-Matrix

To any quiver Q we can associate an skew-symmetric **exchange matrix** $B = B(Q)$. Consider the $n \times 2n$ matrix $\begin{bmatrix} B & I \end{bmatrix}$ and any mutation sequence $\boldsymbol{w} = [i_1, \ldots, i_\ell]$. After the mutations at the indices $i_1, \ldots, i_\ell$ consecutively, we obtain $\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix}$. The matrix $C^{\boldsymbol{w}}$ is known as the C-matrix, and its row vectors are the c-vectors. Every c-vector has either all non-negative or all non-positive entries [DWZ08].

For example, if we mutate the quiver F in Figure 1 at $\boldsymbol{w} = [1, 2, 3]$, then

$$\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 0 & -305 & 28 & -1 & 0 & 0 \\ 305 & 0 & -11 & 55 & 120 & 11 \\ -28 & 11 & 0 & -5 & -11 & -1 \end{array}\right] \tag{1}$$

Main Result

Let Q be a fork with n vertices, and $\boldsymbol{w}$ be a fork-preserving mutation sequence. Then every c-vector of Q obtained from $\boldsymbol{w}$ is a solution to a quadratic equation of the form

$$\sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} \pm q_{ij} x_i x_j = 1, \tag{2}$$

where q_{ij} is the number of arrows between the vertices i and j in Q. For the quiver F in Figure 1, the quadratic equation is given by

$$x^2 + y^2 + z^2 - 3xy - 5xz + 4yz = 1.$$

The row vectors of the above C-matrix satisfy this equation.

Corollary

From the proof of our main result, we found the following corollary, independently discovered by Ahmet Seven.

Let Q be a mutation-cyclic quiver with 3 vertices. Then every c-vector of Q is a solution to a quadratic equation of the form (2) with $n = 3$.

Reflections

When $\boldsymbol{w} = []$, we let r_i be a simple reflection in the universal Coxeter group on n generators, $\mathcal{W}$, for each $i \in \{1, ..., n\}$. For each mutation sequence $\boldsymbol{w}$ and each $i \in \{1, ..., n\}$, define $r_i^{\boldsymbol{w}} \in \mathfrak{R}$ inductively as follows:

$$r_i^{\boldsymbol{w}[k]} = \begin{cases} r_k^{\boldsymbol{w}} r_i^{\boldsymbol{w}} r_k^{\boldsymbol{w}} & \text{if } b_{ik}^{\boldsymbol{w}} c_k^{\boldsymbol{w}} > 0, \\ r_i^{\boldsymbol{w}} & \text{otherwise.} \end{cases}$$

Each $r_i^{\boldsymbol{w}}$ can be written in the form

$$r_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}} s_i (g_i^{\boldsymbol{w}})^{-1}, \quad g_i^{\boldsymbol{w}} \in \mathcal{W}, \quad i \in \{1, ..., n\}.$$

Our quiver F in Figure 1 produces

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2 r_1 r_3 r_1 r_2, \quad r_3^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2.$$

for $\boldsymbol{w} = [1, 2, 3]$.

Coxeter Element

Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, we have

$$r_{\lambda(1)}^{\boldsymbol{w}} \cdots r_{\lambda(n)}^{\boldsymbol{w}} = r_{\rho(1)} \cdots r_{\rho(n)}$$

for some permutations $\lambda, \rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on $\{1, ..., n\}$ and $r_1, ..., r_n$ are the initial reflections. The matrix $B^{\boldsymbol{w}}$ determines λ, and the first mutation of $\boldsymbol{w}$ determines ρ.

The running example produces

$$r_1^{\boldsymbol{w}} r_3^{\boldsymbol{w}} r_2^{\boldsymbol{w}} = r_3 r_1 r_2$$

for $\boldsymbol{w} = [1, 2, 3]$.

Admissible Curves

As a corollary to the above result on reflections, we find that, for each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, there exist pairwise non-crossing and non-self-crossing admissible curves (see [LLM23]) $\eta_i^{\boldsymbol{w}}$ such that $r_i^{\boldsymbol{w}} = \nu(\eta_i^{\boldsymbol{w}})$ for every $i \in \{1, ..., n\}$.

Generalized Intersection Matrix

The **generalized intersection matrix** (GIM), denoted by A, associated to an exchange matrix B is given by a linear ordering $\prec$ of $\{1, \ldots, n\}$ and

$$a_{ij} = \begin{cases} b_{ij} & \text{if } i \prec j, \\ 2 & \text{if } i = j, \\ -b_{ij} & \text{if } i \succ j. \end{cases}$$

L-Matrix

Let $\mathrm{sgn} = \{1, -1\}$ be the group of order 2, and consider the natural group action $\mathrm{sgn} \times \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$. Choose an ordering $\prec$ on $\{1, ..., n\}$ to fix a GIM A associated to an exchange matrix B, and define

$$l_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}}(\alpha_i) \in \mathbb{Z}^n / \mathrm{sgn}, \qquad i \in \{1, ..., n\},$$

where we set $\alpha_1 = (1, 0, ..., 0), ..., \ \alpha_n = (0, ..., 0, 1)$ and $r_i(\alpha_j) = \alpha_j - a_{ji}\alpha_i$. Then the L-**matrix**, $L^{\boldsymbol{w}}$, associated to A is defined to be the $n \times n$ matrix whose i^{th} row is $l_i^{\boldsymbol{w}}$ for $i \in \{1, ..., n\}$. The vectors $l_i^{\boldsymbol{w}}$ are called the l-**vectors** of A. For example, if we take the ordering $2 \prec 1 \prec 3$, the L-matrix for F from Figure 1 and $\boldsymbol{w} = [1, 2, 3]$ gives us

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 55 & 120 & 11 \\ 5 & 11 & 1 \end{bmatrix} \tag{3}$$

Relation Between L-Matrix And C-Matrix

Let Q be a fork with n vertices, and let $\boldsymbol{w}$ be a fork-preserving mutation sequence. For each i and j in $\{1, ..., n\}$, we have that $|l_{ij}^{\boldsymbol{w}}| = |c_{ij}^{\boldsymbol{w}}|$. In other words, the entries of l-vectors are equal to the entries of c-vectors up to sign. Our running example has $l_1^{\boldsymbol{w}} = -c_1^{\boldsymbol{w}}$, $l_2^{\boldsymbol{w}} = c_2^{\boldsymbol{w}}$, and $l_3^{\boldsymbol{w}} = -c_3^{\boldsymbol{w}}$.

References

[DWZ08] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. "Quivers with potentials and their representations I: Mutations". en. In: *Selecta Mathematica* 14.1 (Oct. 2008), pp. 59–119. ISSN: 1420-9020. DOI: `10.1007/s00029-008-0057-9`. (Visited on 11/04/2022).

[LLM23] Kyu-Hwan Lee, Kyungyong Lee, and Matthew R. Mills. "Geometric description of C-vectors and real Lösungen". In: *Mathematische Zeitschrift* 303.2 (Jan. 2023), p. 44. ISSN: 1432-1823. DOI: `10.1007/s00209-022-03180-8`.

[War14] Matthias Warkentin. "Exchange Graphs via Quiver Mutation". en. In: *Dissertation* (Jan. 2014), p. 103. URL: `https://d-nb.info/1214302807/34`.

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