The f-vector of flow polytopes for complete graphs

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Abstract

The Chan-Robbins-Yuen polytope (CRY_n) of order n is a face of the Birkhoff polytope of doubly stochastic matrices that is also a flow polytope of the directed complete graph K_{n+1} with netflow $(1, 0, 0, \ldots, 0, -1)$. We give generating functions and explicit formulas for computing the f-vector of CRY_n as well as any flow polytope of the complete graph having arbitrary (non-negative) netflow vector.

Motivation

The **Chan-Robbins-Yuen polytope** (CRY_n) of order n is defined as the convex hull of n-by-n permutation matrices π for which $\pi_{i,j} = 0$ for $j \geq i+2$. CRY_n has normalized volume equal to the product of the first $n-2$ Catalan numbers (Zeilberger, 1998), though a combinatorial proof of this fact remains elusive. CRY_n is also a face of the Birkhoff polytope of doubly stochastic matrices having dimension $\binom{n}{2}$ and 2^{n-1} vertices, and it is also an example of a flow polytope.

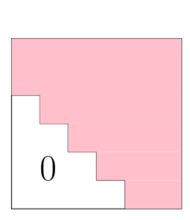

Figure 1: Schematic of the support of matrices appearing in the convex hull of CRY_n.

Flow Polytopes

Fix $G = (V, E)$ a directed, acyclic graph on the vertex set $V = [n+1] = \{1, \ldots, n+1\}$ and $\mathbf{a} = (a_1, \ldots, a_n, -\sum_{i=1}^n a_i)$ a **netflow vector** with each $a_i \in \mathbb{Z}$. Then the **Flow polytope** determined by G and $\mathbf{a}$ is:

$$\mathcal{F}_G(\mathbf{a}) := \{\text{flows } f : E \to \mathbb{R}_{\geq 0} \mid \text{net flow vertex } i \text{ is } a_i\}.$$

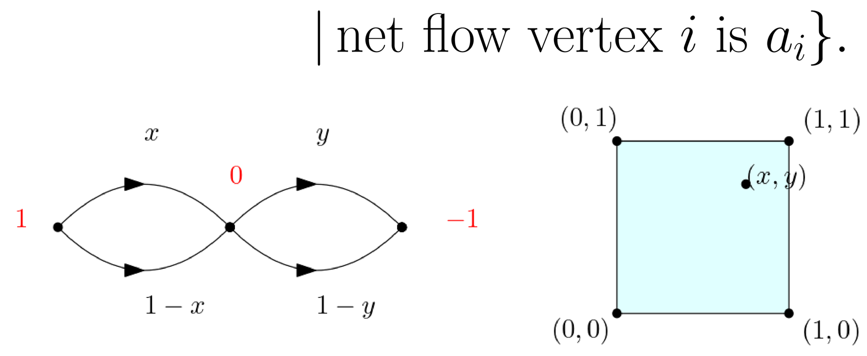

Figure 2: A directed cyclic graph and netflow vector (left) and the corresponding flow polytope (right).

Example:

$$f^{(6)}(1,0,0,1;x) = \frac{1}{x} + \tilde{f}^{(6)}(1,0,0,1;x) + 2\tilde{f}^{(5)}(1,0,1;x) + \tilde{f}^{(3)}(1,1;x)$$

Faces of Flow Polytopes

One particularly powerful use of flow polytopes arises from the following theorem:

Theorem 1. (Hille (2003); Gallo–Sodini (1978)) The d-dimensional faces of $\mathcal{F}_G(\mathbf{a})$ correspond to subgraphs H of G that have **1st Betti number** ($|E| - |V| + c$) equal to d and which are the support of an $\mathbf{a}$-valid flow, where c is the number of connected components of H.

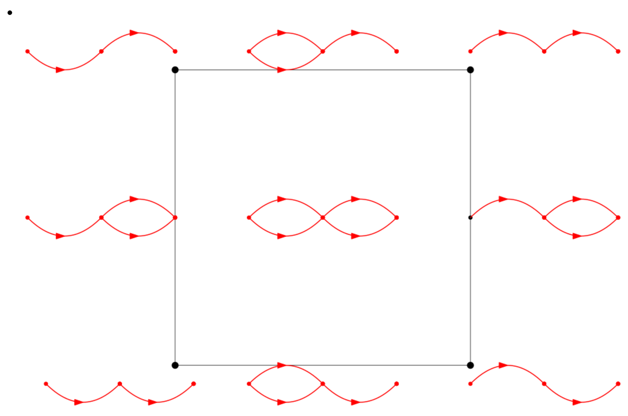

Figure 3: Illustrating Theorem 1.

Andresen–Kjeldsen (1976)

Faces of CRY_n correspond to graphs in the following set:

$$\Omega_n := \{H \subseteq K_{n+1} \mid \text{every } v \in V(H) \text{ lies along a direct path from } v_1 \text{ to } v_{n+1}\}$$

This set was first studied by Andresen–Kjeldsen (1976) working on automata theory. They computed $|\Omega_n|$ by first enumerating the set of **primitive** subgraphs:

$$\Omega'_n := \{H \in \Omega_n \mid V(H) = \{v_1, \ldots, v_{n+1}\} \text{ and } H \text{ is connected}\}.$$

By Theorem 1, the f-vector of CRY_n is a generating function for Ω_n keeping track of $\beta_1(H)$. This leads us to define the notion of *primitive f-vectors*.

Important Results

- We provide a formula for the f-vector for any flow polytope of the complete graph having non-negative netflow vector.
- In particular, we obtain the first-known formula for the f-vector of CRY_n.

$$f^{(n)}(x) = \frac{1}{x} + \frac{1}{x^n} \sum_{m=0}^{n-2} (-1)^m (1+x)^m \pi_{n-m}(x) \cdot h_m((x+1)^1 - 1, \ldots, (x+1)^{n-m-1} - 1).$$

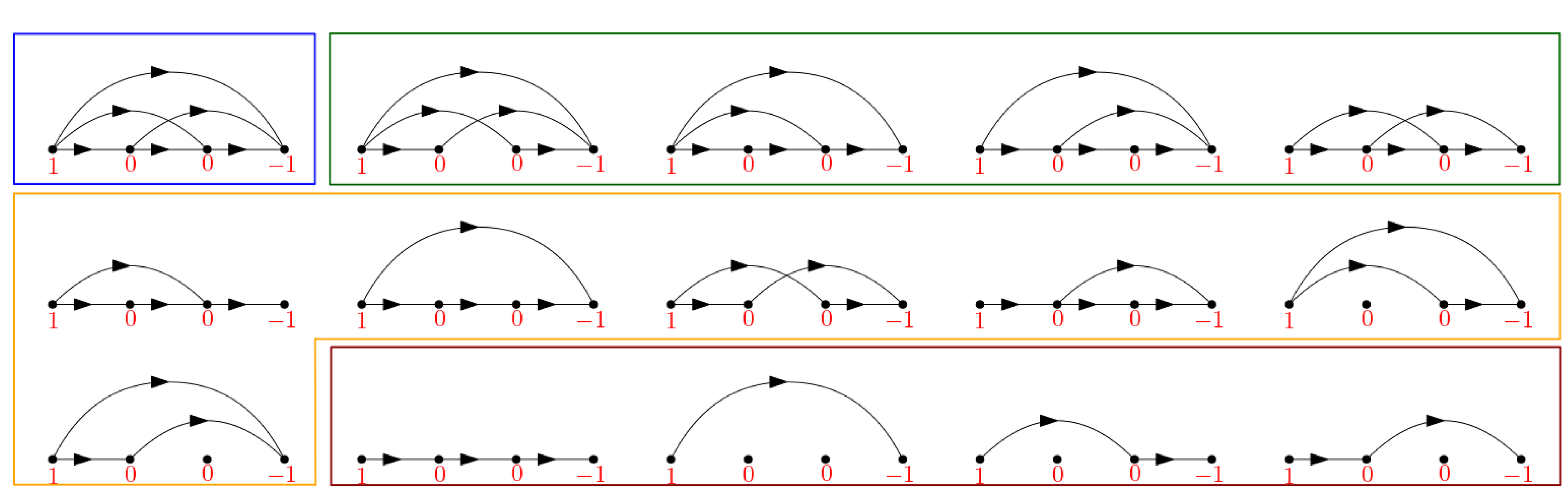

Figure 4: The elements of Ω_3 grouped by first Betti number, corresponding to the f-vector $(1, 4, 6, 4, 1)$ of CRY_3. The primitive f-vector is $(0, 1, 4, 4, 1)$.

Primitive f-vectors

Definition 1. The **primitive f-vector** of $\mathbf{Flow}_n(\mathbf{a})$, denoted $\tilde{f}^{(n)}(\mathbf{a})$ (or as $\tilde{f}^{(n)}(\mathbf{a};x)$ if written as a polynomial) is a generating function over the set of $\mathbf{a}$-valid subgraphs of K_{n+1} that are primitive (use the entire vertex set) keeping track of the first Betti number.

Lemma. For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n:

$$f^{(n)}(\mathbf{a};x) = \frac{1}{x} + \sum_{\mathbf{b} \preceq \mathbf{a}} \tilde{f}^{(|\mathbf{b}|)}(\mathbf{b};x)$$

where $\mathbf{b} \preceq \mathbf{a}$ if $\mathbf{b}$ can be obtained from $\mathbf{a}$ by deleting some subset (possibly empty) of the zeros in $\mathbf{a}$ and where $|\mathbf{b}|$ is the length of $\mathbf{b}$.

Corollary. Entries of the f-vector and primitive f-vector of CRY_n satisfy: $f_d^{(n)} = \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{f}_d^{(n-i)}$.

Quasisymmetric Polynomials

Definition. For α an integer composition of n with $\ell(\alpha)$ parts,

$$P_\alpha(x_1, \ldots, x_n) := \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} \mathbf{x}^\beta$$

where $\mathbf{x}^\beta := x_1^{\beta_1} \cdots x_{\ell(\beta)}^{\beta_{\ell(\beta)}}$, and where the relation $\succeq$ is the standard relation of *refinement* on compositions.

Note that this is very similar to the change-of-basis formula to write a monomial quasisymmetric function in terms of Gessel's *fundamental quasisymmetric functions.*

$$M_\alpha = \sum_{\beta \succeq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta.$$

n	$\tilde{f}$-vector of CRY_n
2	$(0, 1, 1)$
3	$(0, 1, 4, 4, 1)$
4	$(0, 1, 11, 33, 42, 26, 8, 1)$

Reverse Compositions

Every non-negative netflow vector $\mathbf{a}$ determines an integer composition $\text{revcomp}(\mathbf{a})$ as follows.

1. Read the entries of $\mathbf{a}$ from right to left
2. Inductively create a block whenever a new nonzero entry is encountered
3. Return the tuple of sizes coming from the list of blocks

Example: If $\mathbf{a} = (1, 1, 0, 0, 1, 0, 1, 0)$, we get blocks $(0, 1)$, $(0, 1)$, $(0, 0, 1)$, and (1). Hence $\text{revcomp}(\mathbf{a}) = (2, 2, 3, 1)$.

Main Theorems

Theorem. For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n, let α be the composition of n given by $\alpha = \text{revcomp}(\mathbf{a})$. Then the primitive f-vector of $\mathbf{Flow}_n(\mathbf{a})$ written as a polynomial is given by:

$$\tilde{f}^{(n)}(\mathbf{a};x) = \frac{1}{x^n} P_\alpha(x, (x+1)^2 - 1, \ldots, (x+1)^n - 1)$$

Corollary. For CRY_n:

$$\tilde{f}^{(n)}(x) = \frac{1}{x^n} \sum_{m=0}^{n-1} (-1)^m \pi_{n-m}(x) \cdot h_m((x+1)^1 - 1, \ldots, (x+1)^{n-m} - 1)$$

where $\pi_n(x) := x^n [n]_{x+1}! = \prod_{i=1}^n ((x+1)^i - 1)$ and where h_k is a complete homogeneous symmetric polynomial.

Theorem: Let α be the integer composition of n given by $\alpha = \text{revcomp}(\mathbf{a})$. Then the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ written as a Laurent polynomial is given by:

$$f(\mathbf{a};x) = \frac{1}{x} + \frac{1}{x^n} \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \pi_{\ell(\beta)}(x) \cdot \mathbf{x}^{\beta - 1}|_{x_i = (x+1)^i - (x+1)}.$$

n	f-vector of CRY_n
1	$(1, 1)$
2	$(1, 2, 1)$
3	$(1, 4, 6, 4, 1)$
4	$(1, 8, 26, 45, 45, 26, 8, 1)$

Selected References

[1] E. Andresen and K. Kjeldsen. On certain subgraphs of a complete transitively directed graph. *Discrete Mathematics*, 14(2):103–119, 1976.

[2] Lutz Hille. Quivers, cones and polytopes. *Linear Algebra and its Applications*, 365:215–237, 2003.

[3] G. Gallo and C. Sodini. Extreme points and adjacency relationship in the flow polytope. *Calcolo*, 15(3):277–288, 1978.